Taxing Wealth and Capital Income with Heterogeneous Returns*

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Preliminary and incomplete. Please do not circulate.

Abstract

When is a wealth tax preferable to a capital income tax? We study this question in an infinite-horizon model with entrepreneurs and workers, in which entrepreneurial firms are subject to idiosyncratic productivity shocks and collateral constraints. There are two types of steady-state equilibria: one that is efficient and exhibits homogeneous returns, but requires unreasonably high borrowing limits, and a second one that is inefficient and exhibits capital misallocation and heterogeneous returns, which emerges under a wide set of plausible parameter values. In the heterogeneous returns equilibrium, replacing a capital income tax with a wealth tax increases steady-state aggregate productivity and output if (and only if) entrepreneurial productivity is persistent. These gains result from the use-it-or-lose-it effect of wealth taxes, which causes a reallocation of capital from entrepreneurs with low productivity to those with high productivity. We also provide necessary and sufficient conditions for a switch to wealth taxes to imply higher average welfare. These conditions amount to a lower bound on the output elasticity with respect to capital, which is close to $1/3$ for most parameter combinations. We then turn to the optimal taxation problem of a government that can choose any combination of a wealth tax and capital income taxes to maximize welfare. We show that the optimal wealth tax is positive and the capital income tax is negative (a subsidy) when the output elasticity with respect to capital is sufficiently high, the signs flip when the elasticity is sufficiently low, and both taxes are positive in the range between these two thresholds.

JEL Codes: E21, E22, E62, H21.
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1 Introduction

Not all forms of capital taxation are equivalent. Wealth taxes (taxes on the stock of wealth) can improve aggregate productivity, output and welfare when compared with capital income taxes (taxes on the flow of income from wealth). However, these gains are not a necessary feature of all economies. For instance, both forms of taxation are equivalent if all individuals have the same return on wealth.[1] In practice, however, there is ample evidence of return heterogeneity, opening the door for efficiency gains from wealth taxation. In fact, these gains arise in quantitative models with heterogenous returns capable of reproducing the extreme wealth concentration and fast accumulation dynamics observed in the data (Guvenen, Kambourov, Kuruscu, Ocampo and Chen, 2019).

In this paper, we study theoretically the conditions under which replacing capital income taxes with wealth taxes generates efficiency and welfare gains. We do this in a general equilibrium framework with infinitely lived entrepreneurs and workers, in which entrepreneurial firms are subject to idiosyncratic productivity shocks and collateral constraints, in the spirit of Angeletos (2007) and Moll (2014). We show that collateral constraints are binding in equilibrium generating dispersion in entrepreneurial returns, unless the constraints are unreasonably loose. Furthermore, in this equilibrium with return heterogeneity replacing capital income taxes with wealth taxes generates efficiency gains if and only if idiosyncratic productivity shocks are persistent (but not permanent). Under this condition the use-it-or-lose-it effect of wealth taxes operates, reallocating capital towards more productive entrepreneurs, in turn increasing aggregate productivity, output and wages as in Guvenen et al. (2019). Finally, we show that the welfare gains from wealth taxation depend on the capital-elasticity of output. A higher elasticity increases the positive effect of wealth taxes on wages by making them more responsive to (efficiency–adjusted) capital.

Entrepreneurs in the model derive their income from selling final goods in a perfectly competitive market. All entrepreneurs produce using the same constant-returns-to-scale technology that combines capital and labor but they differ in their productivity, which changes stochastically and independently across entrepreneurs. We say that productivity is persistent if there is any degree of positive autocorrelation in the entrepreneurs' productivity

[1] To see this, let r be the rate of return, τ_a the tax rate on wealth, and τ_k the tax rate on capital income. The after tax wealth of an individual with initial wealth a_i is $(1-\tau_a)\,a_i + ra_i$ under wealth taxes and $a_i + (1-\tau_k)\,ra_i$ under capital income taxes. Both expressions coincide for all agents if $\tau_a = r\tau_k$, and in that case the behavior of agents also coincides and both tax systems raise the same revenue.

process. There is a bond market, with zero net supply, through which entrepreneurs can borrow from each other subject to a borrowing limit. Entrepreneurs with high productivity borrow to invest in their own firms, while those with low productivity partially invest in their own firm and/or lend their remaining wealth. Workers are hand-to-mouth and consume their wages, so that all the wealth in the economy is held by entrepreneurs.

We show the following main results. First, we prove that two types of steady-state equilibria with different properties can emerge across the parameter space. The first steady-state equilibrium is efficient but requires high borrowing limits. In this equilibrium the most productive entrepreneurs employ all the capital in the economy, and all entrepreneurs earn the same return. The second equilibrium is inefficient and exhibits capital misallocation. In this equilibrium collateral constraints bind for more productive entrepreneurs, who then earn higher returns on wealth than less productive ones. The presence of return heterogeneity makes this the empirically relevant equilibrium. Moreover this steady-state equilibrium emerges under a wide set of plausible parameter values, and, as a result, we focus only on this equilibrium in the rest of the paper. We explicitly solve for the maximum debt-to-wealth ratio—that characterizes the borrowing limit—needed for the heterogeneous returns equilibrium to prevail as a function of the model's parameters and show that it implies debt-to-GDP ratios larger than the U.S. levels. Thus, generating return heterogeneity in equilibrium does not require imposing strict borrowing limits to entrepreneurs.

Second, we show that a marginal increase in wealth taxes that replace capital income taxes increases aggregate efficiency (TFP), capital, output, and wages if and only if entrepreneurial productivity is persistent and the economy is in the heterogeneous returns equilibrium. Under capital income taxation, entrepreneurs who are more productive, and therefore generate more income, pay higher taxes. Under wealth taxation, entrepreneurs who have similar wealth levels pay similar taxes regardless of their productivity. Since the tax reform reduces the tax burden of more productive entrepreneurs, it allows them to accumulate more wealth, this is the use-it-or-lose-it effect. As a result, when entrepreneurial productivity is persistent, more productive entrepreneurs in the future receive more wealth under wealth taxes. This increase in the share of wealth held by more productive entrepreneurs increases aggregate productivity, output, and wages. Moreover, the average after-tax return goes down and the dispersion in after-tax rates of returns increases as a direct consequence of this reallocation.

Third, we study the welfare implications of a switch from a capital income tax to a wealth tax and provide necessary and sufficient conditions under which the switch leads to an increase in welfare. Workers unambiguously benefit from higher wealth taxes through the rise in wage. The consequences for entrepreneurial welfare depend on the response of after-tax returns. The decrease in expected returns hurts the payoffs of all entrepreneurs and ultimately leads to welfare losses for the entrepreneurs as a group. Thus, the aggregate welfare gains of an increase in wealth taxes depend on the strength of the increase in wages which is linked to the output elasticity with respect to capital. Indeed, we show that the conditions for welfare gains amount to a lower bound on the output elasticity with respect to capital, which is close to one-third for most parameters. We also consider alternative welfare measures that take into account the changes in wealth accumulation by entrepreneurs. Higher wealth taxes increase overall wealth and thus ameliorate (and potentially overturn) the welfare loss of entrepreneurs. Consequently, taking into account wealth accumulation relaxes the conditions for welfare gains from wealth taxation.

Fourth, we study the optimal combination of capital income and wealth taxes and derive the optimal tax formula as a function of output/wage elasticity with respect to capital as well as other parameters. The optimal tax weighs the benefit of higher wealth taxes for workers from the increase in wages to the cost to entrepreneurs from lower expected returns. With a larger elasticity, the response of wages to the increases in TFP is higher, which in turn benefits the workers who are the majority of the population, and as a result the optimal wealth tax turns out to be higher. Overall, we characterize the optimal taxes as functions of a lower bound and an upper bound on the elasticity. If the elasticity is above the upper bound, the optimal wealth tax is positive and the capital income tax is negative (a subsidy), the signs flip when the elasticity is lower than the lower bound, and both taxes are positive in the narrow range between these two thresholds.

We also consider three separate extensions to our framework: introducing a corporate sector, incorporating entrepreneurial effort in the entrepreneurs' production function, and incorporating excess returns in the sense that higher returns do not necessarily imply higher entrepreneurial productivity. Although the main points in the analysis remain unchanged, these extensions are informative of the main mechanisms operating in the model and highlight the overall appeal of our theoretical framework.

Finally, we study an alternative perpetual-youth model where all entrepreneurs survive to the next period with a certain constant probability. There are no annuity markets and total accidental bequests are distributed equally among the newborn. Each newborn entrepreneur draws a productivity type, which are permanent over their life span. This alternative model exhibits a stationary wealth distribution across agents that we solve for in closed form. We show that all results from our benchmark model carry on, except that efficiency gains from wealth taxation arise always since types are permanent. We also characterize the effects on wealth inequality. In particular, the top wealth shares increase after an increase in wealth taxes and high-productivity entrepreneurs unambiguously benefit from a shift from capital income to wealth taxes.

Related literature (to be completed)

There is a large literature on capital income taxation (see Chari and Kehoe (1999), Golosov, Tsyvinski and Werning (2006), and Stantcheva (2020) for excellent reviews of the literature).

Our paper is motivated by recent empirical and theoretical evidence that provide strong support for rate of return heterogeneity. First, tracking millions of individuals over long periods of time Fagereng, Guiso, Malacrino and Pistaferri (2020), Bach, Calvet and Sodini (2018), and Smith, Yagan, Zidar and Zwick (2019) document large and persistent differences in rates of return across individuals, even after adjusting for risk and other factors. Second, an active literature on power law models shows that rate of return heterogeneity is key for generating important features of inequality that have proved challenging to explain through other mechanisms, such as the thick Pareto tail of the wealth distribution and the *dynamics* of wealth inequality over time.[2]

2 Benchmark Model

Time is discrete. There are two types of infinitely-lived agents: homogenous workers of size L and heterogenous entrepreneurs of size 2. Workers supply one unit of labor inelastically, behave as hand to mouth agents, and hold no wealth. Workers' and entrepreneurs' preferences take the form $E_0 \sum_{t=1}^{\infty} \beta^{t-1} \log(c_t)$, where $0 < \beta < 1$.

Each entrepreneur produces a homogenous good combining capital k and labor n using

[2]See Gabaix (2009) and Benhabib and Bisin (2018) for excellent reviews of this literature. See also Benhabib, Bisin and Zhu (2011); Benhabib, Bisin and Luo (2017) for the study of the Pareto tail and Gabaix, Lasry, Lions and Moll, 2016; Jones and Kim, 2018 for the dynamics of wealth inequality.

a constant-returns-to-scale technology:[3]

$$y = (zk)^{\alpha} n^{1-\alpha}. \tag{1}$$

We assume that capital does not depreciate for simplicity.

Entrepreneurs differ in their productivity z, which takes values $\{z_l, z_h\}$, where $z_h > z_l \geq 0$, and follows a Markov process with transition matrix

$$\mathbb{P} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} \tag{2}$$

with $0 < p < 1$ being the probability that an individual entrepreneur retains their productivity across periods. The autocorrelation coefficient of the entrepreneurial productivity process is $\rho = 2p - 1$, so that if $p < 1/2$ the productivities are negatively correlated across periods, if $p = 1/2$ the productivities are uncorrelated (entrepreneurial productivity is iid), and if $p > 1/2$ the productivities are persistent across time and positively correlated. This productivity process ensures that there is always a mass 1 of high-type entrepreneurs with productivity z_h and a mass 1 of low-type entrepreneurs with productivity z_l.

Entrepreneurs hire labor at a wage w and can borrow through a bond market at an interest rate r to invest in their firm over and above their own wealth a. Both markets are perfectly competitive. The same bonds, which are in zero net supply, can also be used as a savings device, which will be optimal for entrepreneurs whose return is lower than the interest rate r. Thus, k can be greater or smaller than a.

Entrepreneurs are subject to a collateral constraint. For an entrepreneur who enters a period with a units of wealth, the collateral constraint is

$$k \leq \lambda a, \tag{3}$$

where $\lambda \geq 1$. If $\lambda = 1$ an entrepreneur can use only their wealth in production.

The government levies taxes on capital income at rate τ_k and (beginning-of-period) wealth at rate τ_a to finance an exogenous expenditure G.

[3]This model is equivalent to a model where a final good producer rents labor L from households and perfectly substitutable intermediate inputs produced by entrepreneurs using only capital, where each entrepreneur's output is given by $z_i k_i$. Thus, the entrepreneurial sector is a special case of the one analyzed in Guvenen et al. (2019).

2.1 Entrepreneur's problem

We now summarize the solution to the entrepreneur's problem. The problem is standard and we relegate most details to Appendix A. We start with an entrepreneur's choice of capital and labor to maximize their entrepreneurial income:

$$\Pi^\star(z,a) = \max_{k\leq\lambda a, n\geq 0} (zk)^\alpha n^{1-\alpha} - rk - wn. \tag{4}$$

The constant-return-to-scale technology with which the entrepreneur produces implies that entrepreneurs whose marginal return to capital is greater than the interest rate borrow up to the limit and set $k^\star = \lambda a$ and those whose return is below the interest rate do not produce and instead earn the return r in the bond market on wealth a. Consequently, the optimal entrepreneurial income can be written as

$$\Pi^\star(z,a) = \pi^\star(z)\,a \quad = \quad \begin{cases} \left(\alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z - r\right)\lambda a & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z > r \\ 0 & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z \leq r, \end{cases} \tag{5}$$

where $\pi^*(z)$ is the excess return an entrepreneur earns above the interest rate r.

We now turn to the optimal savings problem of the entrepreneur, which amounts to solving the following dynamic programming problem:

$$V(a,z) = \max_{a'} \log(c) + \beta\sum_{z'}\mathbb{P}(z' \mid z)\,V(a',z') \tag{6}$$
$$\text{s.t. } c + a' = R(z)\,a,$$

where $R(z) \equiv (1-\tau_a) + (1-\tau_k)(r + \pi^\star(z))$ is the gross return on savings. In solving this problem we take as given time-invariant taxes τ_a and τ_k, and prices r and w, anticipating the results of Section 2.2 below. The solution is the following optimal savings rule:

$$a' \quad = \quad \beta R(z)\,a. \tag{7}$$

The lifetime value of an entrepreneur with assets a and current productivity z_i, $i \in \{l,h\}$, is

$$V_i(a) = m_i + \frac{1}{1-\beta}\log(a), \tag{8}$$

with $m_i = \frac{\log(1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2}\log(\beta) + \frac{(1-\beta)\log R_i + \beta(1-p)(\log R_l + \log R_h)}{(1-\beta)^2(1-\beta(2p-1))}$ and $R_i \equiv R(z_i)$. We present the details for this result in Appendix A. As for workers, they hold no assets and consume all of their labor income, hence, their lifetime utility is simply:

$$V_w = \frac{1}{1-\beta} \log w. \tag{9}$$

2.2 Equilibrium

We focus on the equilibrium values for aggregate quantities and prices of the economy. We group entrepreneurs by their productivity and use each group's aggregate wealth to analyze the equilibrium. For this purpose, let A_i be the aggregate wealth held by type $i \in \{h, l\}$ entrepreneurs at the beginning of the period. This is made possible because labor demand and savings decision rules are linear in wealth with the same coefficient within each productivity type.

The equilibrium interest rate must be in between the marginal return to capital of the low and high productivity entrepreneur at any point in time, i.e.,

$$\alpha \left(\frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} z_l \quad \leq r \leq \quad \alpha \left(\frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} z_h. \tag{10}$$

Outside of this range, the capital markets would not clear. In fact, the equilibrium interest rate must equal one of the two bounds depending on the net demand for assets in the economy. If $(\lambda - 1) A_h > A_l$, the high-productivity entrepreneurs demand more funds than can be supplied by low types and bid up the equilibrium interest rate to their marginal product (the upper bound in equation 10). If this is the case, all entrepreneurs earn the same rate of return and the equilibrium allocation coincides with the (efficient) complete markets allocation. Moreover, capital income and wealth taxes are equivalent.

If $(\lambda - 1) A_h < A_l$, there are more funds available in the hands of low-productivity entrepreneurs than the amount demanded by high types, so that the equilibrium interest rate is bid down to the marginal product of the low-productivity entrepreneurs:

$$r = \alpha \left(\frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} z_l. \tag{11}$$

In this case, high-productivity entrepreneurs are collateral constrained and their aggregate capital demand is $K_h = \lambda A_h$. Low types are indifferent between investing in their firm or lending in the bond market. Their capital demand is determined as the residual in the capital market clearing condition: $K_l = A_l - (\lambda - 1) A_h$. This equilibrium features return heterogeneity, with $R_h > R_l$, misallocation of capital, and is inefficient (relative to the first best allocation). Most importantly, capital income and wealth taxes are not equivalent. Thus, we will focus on this case, which we refer to as the heterogeneous returns equilibrium. We provide a necessary and sufficient condition for $(\lambda - 1) A_h < A_l$ to hold in steady state in proposition 1, and show that this condition is satisfied under a wide range of parameter values.

In the heterogeneous return equilibrium the productivity of capital is endogenous and plays a crucial role in determining the aggregate variables of the economy. In anticipation of these results we define the following:

Definition 1. We define the wealth-weighted productivity of capital as

$$Z \equiv s_h z_\lambda + (1 - s_h) z_l \tag{12}$$

where $z_\lambda \equiv z_h + (\lambda - 1)(z_h - z_l)$ denotes the effective productivity of high-productivity entrepreneurs and $s_h \equiv {}^{A_h}/_K$ denotes the wealth share of high-productivity entrepreneurs.

Notice that z_λ reflects the return high-productivity entrepreneurs earn from their own wealth, captured by z_h, and the excess return from borrowed capital, $(\lambda - 1)(z_h - z_l)$.

We can now solve for the aggregate variables that characterize the heterogeneous return equilibrium as a function of the aggregate capital K and productivity Z. This is possible because of the constant-returns-to-scale technology that the entrepreneurs operate. We gather the relevant expressions in the following Lemma:

Lemma 1. ***(Aggregate Variables in Equilibrium)*** *In the heterogenous return equilibrium* $((\lambda - 1) A_h < A_l)$*, output, wages, interest rate, and gross returns on savings are:*

$$Y = (ZK)^\alpha L^{1-\alpha} \tag{13}$$

$$w = (1 - \alpha) \left({}^{ZK}/_L\right)^\alpha \tag{14}$$

$$r = \alpha \left({}^{ZK}/_L\right)^{\alpha-1} z_l \tag{15}$$

$$R_l = (1 - \tau_a) + (1 - \tau_k) \alpha \left({}^{ZK}/_L\right)^{\alpha-1} z_l \tag{16}$$

$$R_h = (1 - \tau_a) + (1 - \tau_k) \alpha \left({}^{ZK}/_L\right)^{\alpha-1} z_\lambda. \tag{17}$$

The aggregation results above make it clear that Z is an endogenous measure of productivity that depends on the wealth shares of entrepreneurs. Since the second equilibrium is efficient, the productivity of capital in that economy is $Z^{\star} = z_h$. Then the TFP loss due to the financial friction and the resulting misallocation of capital is given as the ratio $\frac{TFP^{\star}}{TFP} = \left(\frac{z_h}{s_h z_\lambda + (1-s_h) z_l}\right)^{\alpha}$, which declines with the wealth share of h-type s_h and the borrowing limit λ, and increases with the productivity gap $\frac{z_h}{z_l}$.

Remark. The aggregate production function can be rewritten in a familiarly Cobb-Douglas form defining $Q \equiv ZK$ as the effective capital as in Guvenen et al. (2019):[4]

$$Y = Q^{\alpha} L^{1-\alpha}.$$

Evolution of aggregates: Using the savings decision rules of each type, we can obtain the law of motions for aggregate wealth held by each type as

$$A_h^{'} \;=\; p\beta R_h A_h + (1-p)\,\beta R_l A_l \quad \text{and} \quad A_l^{'} = (1-p)\,\beta R_h A_h + p\beta R_l A_l. \tag{18}$$

Then the law of motion for aggregate wealth/capital ($K \equiv A_l + A_h$) becomes

$$K^{'} \;=\; \beta\,(1-\tau_a)\,K + \beta\,(1-\tau_k)\,\alpha\,(ZK)^{\alpha}\,L^{1-\alpha}. \tag{19}$$

The law of motion for Q follows from $Q' = z_\lambda A_h' + z_l A_l'$ after substituting A_h' and A_l' from equation (18):

$$\begin{aligned} Q^{'} = {} & \beta\,(1-\tau_a)\left(\rho Q + \frac{z_l + z_\lambda}{2}\,(1-\rho)\,K\right) \\ & + \beta\,(1-\tau_k)\,\alpha\,({}^{Q}\!/_{L})^{\alpha-1}\left(\frac{z_l + z_\lambda}{2}\,(1+\rho)\,Q - z_l z_\lambda \rho K\right). \end{aligned} \tag{20}$$

2.3 Steady State

The steady state of the economy is defined as the state where aggregates A_h, A_l, K, and Z are constant given some constant tax rates τ_a and τ_k. Before reaching the main result stating a necessary and sufficient condition for the existence and uniqueness of the steady state, we provide some intermediate results derived from the evolution of aggregates. First,

[4] The effective capital Q is the productivity-weighted capital $Q = z_\lambda A_h + z_l A_l = z_h K_h + z_l K_l$.

imposing steady state on equation (19), we obtain

$$(1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha}\left({}^{K}\!/_{L}\right)^{\alpha-1} = \frac{1}{\beta}. \tag{21}$$

This equation states that the after-tax gross return on capital should be equal to one over the discount factor in the steady state. Equation (21) gives the aggregate capital K given Z as in the standard growth model except that Z is endogenous here. Our analysis will center on Z since we can analyze all aggregate variables by focusing on Z. For instance, we can write the steady state levels of R_l and R_h as

$$R_h = (1-\tau_a) + \left(\frac{1}{\beta} - (1-\tau_a)\right)\frac{z_\lambda}{Z} \quad \text{and} \quad R_l = (1-\tau_a) + \left(\frac{1}{\beta} - (1-\tau_a)\right)\frac{z_l}{Z}. \tag{22}$$

Finally, inserting equation $Q = ZK$ into equation (20), using equation (21), and imposing steady state we obtain:

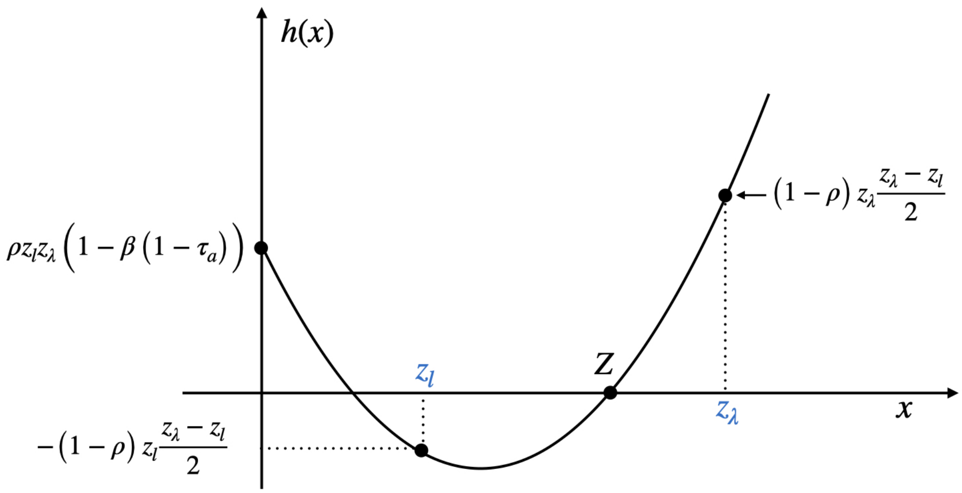
$$(1-\rho\beta(1-\tau_a))\,Z^2 - \frac{z_l + z_\lambda}{2}\,(1+\rho(1-2\beta(1-\tau_a)))\,Z + z_l z_\lambda \rho\,(1-\beta(1-\tau_a)) = 0. \tag{23}$$

This is a quadratic equation in Z that characterizes the steady state of the model. Studying this quadratic equation, we are able to show that there exists a unique steady state and obtain a necessary and sufficient condition for the steady state to feature heterogeneous returns.

Before providing the formal statement of our result in Proposition 1, we discuss the logic behind the proof. Existence and uniqueness of the steady state follow from analyzing the solution to equation (23). It is easy to verify that the first two coefficients, $1-\rho\beta(1-\tau_a)$ and $1+\rho(1-2\beta(1-\tau_a))$, are both positive. For $\rho < 0$ there is a unique solution. On the other hand, for $\rho > 0$ there are two positive roots as illustrated in Figure 1. However, only the larger root satisfies $z_l < Z < z_\lambda$, which has to hold in equilibrium. The steps that lead to the necessary and sufficient condition for the steady state to feature heterogenous returns are more involved, but they boil down to realizing that return heterogeneity $(R_h > R_l)$ necessarily implies that the TFP of capital (Z) is below its efficient level z_h. We can obtain an upper bound $\left(\overline{\lambda}\right)$ on the borrowing constraint parameter λ that guarantees that $Z < z_h$. This upper bound turns out to be not only sufficient but also necessary. Proposition 1 presents the value of $\overline{\lambda}$.

Proposition 1. ***(Existence and Uniqueness of Steady State)*** *There exists a unique*

Figure 1: Steady State Productivity (Z)



Note: *The figure plots* $h(x) = (1 - \rho\beta(1-\tau_a))\, x^2 - {}^{(z_l+z_\lambda)}/_2 \,(1 + \rho(1 - 2\beta(1-\tau_a)))\, x + z_l z_\lambda \rho (1 - \beta(1-\tau_a))$. *The steady state productivity corresponds to the larger root of h, marked with a circle on the horizontal axis.*

steady state that features heterogenous returns ($R_h > R_l$) *if and only if*

$$\lambda < \overline{\lambda} \equiv 1 + \frac{(1-\rho)}{1 + \rho\left(1 - 2\left(\beta(1-\tau_a) + (1 - \beta(1-\tau_a))\frac{z_l}{z_h}\right)\right)}.$$

Corollary 1. *The condition for the steady state to feature heterogeneous returns can be restated as an upper bound on wealth taxes given the values of the other parameters:*

$$\lambda < \overline{\lambda} \longleftrightarrow \tau_a \leq \overline{\tau}_a \equiv 1 - \frac{1}{\beta\left(1 - \frac{z_l}{z_h}\right)}\left[\frac{(\lambda-1)(\rho+1) - (1-\rho)}{2(\lambda-1)\rho} - \frac{z_l}{z_h}\right].$$

We assume that $\lambda < \overline{\lambda}$ in the rest of the analysis, which guarantees that the steady state features return heterogeneity. Corollary 1 gives us an equivalent condition in terms of the level of wealth taxes, $\tau_a < \overline{\tau}_a$. Intuitively, neither λ nor τ_a can bee too high for there to be heterogeneous returns because they both work by reducing the level of misallocation in the economy, λ by loosening the collateral constraint of entrepreneurs and τ_a by increasing the dispersion of (after-tax) returns, as we show in Proposition 2 below. Even though both $\overline{\lambda}$ and $\overline{\tau}_a$ contain the same information, it will prove instructive to treat them separately. We can focus first on the behavior of $\overline{\lambda}$ and then fix $\lambda < \overline{\lambda}\,(\tau_a = 0)$ in our analysis, so that the steady state of the economy without wealth taxes features return heterogeneity. Finally, we will analyze what happens to the economy as we change wealth taxes for $\tau_a < \overline{\tau}_a$ without further changes to λ.

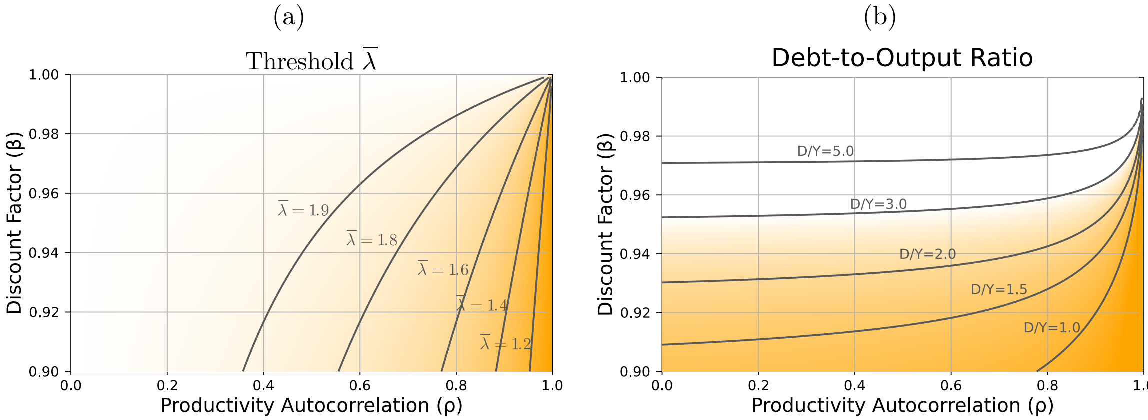
Now, we discuss some properties of $\overline{\lambda}$ and we show that the value of $\overline{\lambda}(\tau_a = 0)$ generates sensible levels of debt in the model. We focus on the case in which productivity is autocorrelated, i.e., $\rho > 0$, because that implies tighter bounds on λ. In fact, it is easy to show that $\overline{\lambda} < 2$ if and only if $\rho > 0$. We can show that $\frac{d\overline{\lambda}}{d\beta} > 0$ so the upper bound get looser if the saving rate of agents goes up (as both low- and high-types accumulate more assets); in contrast, $\frac{d\overline{\lambda}}{d\rho} < 0$ and $\frac{d\overline{\lambda}}{d(z_l/z_h)} > 0$, with the upper bound getting tighter if the persistence of productivity increases (which benefits high-types who would accumulate more wealth) or if the productivity gap between types widens (which could also increase the wealth gap).

We now show that the value of $\overline{\lambda}$ for an economy without wealth taxes allows for sensible levels of borrowing in the economy. One way to gauge this is to compare the entrepreneurial debt-to-GDP ratio from the model when we set $\lambda = \overline{\lambda}$ to the ratio in the data. Guvenen et al. (2019) computes this ratio for the US as 1.52 taking the the sum of non-financial business liability (\$22.79 trillion) and the capitalized value of external funds raised through IPOs and equity issues by US non-financial businesses (\$4.14 trillion) for 2015.

In Figure 2 we report $\overline{\lambda}(\tau_a = 0)$ and the debt-to-GDP ratio $\left({}^{(\lambda-1)A_h}/_{Y}\right)$ for different β and p values if the borrowing limit λ was set to $\overline{\lambda}$. Figure 2a confirms the comparative statics results with respect to β and p. The debt-to-GDP ratio moves in the same direction as $\overline{\lambda}$ and increases markedly with β as expected (Figure 2b). If we take the model period as one year, focusing on the cases where $\beta > 0.94$ seems reasonable. The bottomline is that the debt-to-GDP ratio associated with the $\overline{\lambda}$ limit is typically much higher than the data counterpart of 1.52 except for very high p and low β values. For example, for $\beta = 0.96$ and $p = 0.95$, $\overline{\lambda} = 1.68$ and the debt-to-GDP ratio is 290 percent. Thus, $\overline{\lambda}$ needed for the first equilibrium is not restrictive at all. These results are not driven by unreasonable dispersion in returns either. The return gap between high- and low-productivity entrepreneurs, $R_h - R_l$, is between 2 and 10 percentage points for the relevant combinations of parameters, as we show in Figure D.2 of Appendix D.

As for the properties of the bound on wealth taxes, $\overline{\tau}_a$, the bound gets tighter as either λ or ρ increase. The reason is the same in the discussion above, namely that higher λ or ρ reduce the misallocation of capital either by loosening the collateral constraint of entrepreneurs or by making their productivity more persistent in turn allowing the to save their way out of the constraints (as in Moll, 2014). For high enough λ and ρ the bound turns

Figure 2: Conditions for Steady State with Heterogeneous Returns



Note: *Figure 2a reports the value of $\overline{\lambda}$ found in Proposition 1 for combinations of the autocorrelation of productivity (ρ) and the discount factor (β). The steady state exhibits heterogeneous returns if and only if $\lambda \leq \overline{\lambda}$. Figure 2b reports the debt-to-output ratio when $\lambda = \overline{\lambda}$ computed as $(\overline{\lambda}-1)A_h/Y$. In both figures we set the remaining parameters as follows: $z_l = 0$, $z_h = 2$, $\tau_k = 25\%$, and $\alpha = 0.4$.*

negative and wealth subsidies are needed to reduce the dispersion of of after-tax returns. Figure D.3 in Appendix D illustrates the behavior of $\overline{\tau}_a$ for different combinations of the autocorrelation of productivity, ρ, and λ, along with the debt-to-output-ratio implied by each combination of parameters.

Finally, we would like to discuss another property of the steady state equilibrium when $\lambda < \overline{\lambda}$. In this case, high-productivity entrepreneurs accumulate wealth, while low-productivity entrepreneurs dissave, that is $\beta R_h > 1 > \beta R_l$. This is to be expected because returns satisfy $R_h > R_l$. Moreover, the high-productivity entrepreneurs hold most of the wealth if and only if productivity is autocorrelated ($\rho > 0$). This is a consequence of the saving behavior of entrepreneurs which leads to accumulation of wealth only when they can retain their type across periods. The proof of these results follows directly from the proof of Proposition 1 and we formalize it in Corollary 2.

Corollary 2. ***(Savings Rates and Wealth Shares)*** *For all $\tau_a < \overline{\tau}_a$, the steady state savings rate of type h entrepreneurs is positive and the savings rate of type l entrepreneurs is negative: $\beta R_h > 1 > \beta R_l$. Furthermore, $s_h > 1/2$ if and only if $\rho > 0$.*

2.4 Government Budget Constraint

The government uses capital income and wealth tax revenues to finance non-productive government expenditures G. Thus, the government's budget constraint is

$$G = \tau_k \alpha Y + \tau_a K.$$

In steady state we can simplify this expression by substituting equation (21) to obtain

$$G = \left(\tau_k + \tau_a \frac{\beta (1 - \tau_k)}{1 - \beta (1 - \tau_a)} \right) \alpha Y.$$

Next, we make an assumption that greatly simplifies our upcoming analysis.

Assumption 1. *G is a constant fraction $\theta\alpha$ of aggregate output: $G = \theta \alpha Y$.*

We want to make two remarks here. First, Assumption 1 allows total tax revenue and government expenditure to increase with the size of the economy. However, we will present conditions under which increasing wealth taxes delivers a higher output and average welfare than the capital income tax, while increasing the government's tax revenue. Thus, we could have achieved even a higher output and average welfare from wealth taxes if we had imposed revenue neutrality. Second, this assumption implies a tight link between τ_k and τ_a. With both taxes present, the government budget constraint simplifies to[5]

$$\frac{1 - \tau_k}{1 - \beta (1 - \tau_a)} = \frac{1 - \theta}{1 - \beta}. \tag{24}$$

2.5 Aggregate Variables in Steady State

We now summarize the relationship between the different aggregates and the steady state value of Z. Once the steady state value of Z is known, we can immediately recover the values of s_h, R_h, and R_l. Moreover, we can sign the response of these aggregates to changes in Z. We can go further and recover the values of aggregate capital K, output Y, wages w and the level of assets of each entrepreneurial type A_h and A_l under Assumption 1. This will payoff in the next section when we turn to the effect of taxes on the steady state of the model. From equation (23) it is clear that the steady state value of Z depends on only τ_a, and under Assumption 1 the relationship between Z and the other aggregates is

[5] $\tau_k = \theta$ if there is only capital income tax ($\tau_a = 0$) and $\tau_a = \frac{\theta(1-\beta)}{\beta(1-\theta)}$ if there is only wealth tax ($\tau_k = 0$).

independent of the combination of taxes, instead depending on the size of the government budget, captured by the parameter θ. Together, these results will allow us to study the effects of continuous changes in wealth taxes.

Recall that $Z = s_h z_\lambda + (1 - s_h) z_l$ is the measured TFP of capital in the economy. Then, an increase in TFP necessarily implies an increase in the wealth share of the high-productivity entrepreneurs, s_h. This link between productivity and the distribution of assets also lets us characterize the steady state rates of return R_h and R_l in terms of productivity using the evolution equations for assets (18). We show that the increase in the wealth share of high-productivity entrepreneurs reflects an increase in the dispersion of returns. In fact, the returns of high-productivity entrepreneurs increase while the returns of low-productivity entrepreneurs decrease. Average returns go down. Importantly, these results do not depend on how (or whether) the government's budget is balanced.

Lemma 2. ***(Wealth Shares and Returns in Steady State)*** *For all $\tau_a < \overline{\tau}_a$, the following equations and inequalities hold in steady state:*

$$s_h = \frac{1 - \beta R_l}{\beta (R_h - R_l)} = \frac{Z - z_l}{z_\lambda - z_l} \qquad \frac{ds_h}{dZ} = \frac{1}{z_\lambda - z_l} > 0 \tag{25}$$

$$R_h = \frac{1}{\beta (2p - 1)} \left(1 - \frac{1 - p}{s_h}\right) \qquad \frac{dR_h}{dZ} > 0 \tag{26}$$

$$R_l = \frac{1}{\beta (2p - 1)} \left(1 - \frac{1 - p}{1 - s_h}\right) \qquad \frac{dR_h}{dZ} < 0. \tag{27}$$

Moreover, the average returns are always decreasing with productivity, $\frac{d(R_l + R_h)}{dZ} < 0$, while the geometric average of returns decreases if and only if $\rho > 0$, $\frac{d(R_h R_l)}{dZ} < 0$.

Finally, we turn to determining the effects of changes in Z on aggregate capital and output. An increase in Z is an increase in productivity, it is then natural that aggregate capital K and effective capital Q are increasing in Z. The effect on output Y and wages w follow immediately. In order to show this we rely on Assumption 1 which makes it possible to express the steady state value of aggregates in terms of the size of government spending (captured by θ) independently of the combination of taxes used by the government. Finally, we show that A_h increases in Z and that the response of A_l depends on the elasticity of output with respect to capital. We group all these results in the following Lemma:

Lemma 3. ***(Aggregate Variables in Steady State)*** *If $\tau < \overline{\tau}_a$ and under Assumption 1,* *the steady state level of aggregate capital is*

$$K = \left(\frac{\alpha\beta\,(1-\theta)}{1-\beta} \right)^{\frac{1}{1-\alpha}} L Z^{\frac{\alpha}{1-\alpha}} \tag{28}$$

and the steady state elasticities of aggregate variables with respect to productivity are

$$\xi_K = \xi_Y = \xi_w = \xi \equiv \frac{\alpha}{1-\alpha} \qquad \text{and} \qquad \xi_Q = 1 + \xi, \tag{29}$$

where $\xi_x \equiv \frac{d \log x}{d \log Z}$ *is the elasticity of variable* x *with respect to* Z*. Effective capital is* $Q = ZK$*, aggregate output is* $Y = Q^{\alpha} L^{1-\alpha}$*, and wage is* $w = (1-\alpha)\,Y$ *from Lemma 1. Moreover, the wealth levels of each entrepreneurial type in steady state are*

$$A_h = \frac{Z - z_l}{z_\lambda - z_l} K \qquad\qquad \frac{dA_h}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}} \left(Z - \alpha z_l \right) > 0 \tag{30}$$

$$A_l = \frac{z_\lambda - Z}{z_\lambda - z_l} K \qquad\qquad \frac{dA_l}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}} \left(\alpha z_\lambda - Z \right), \tag{31}$$

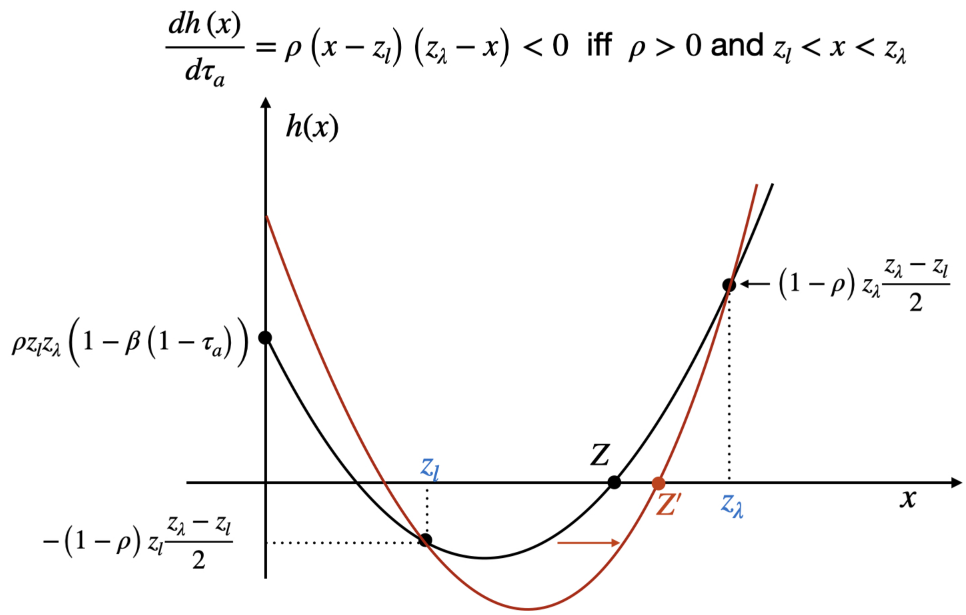
where $\frac{dA_l}{dZ} < 0$ *if and only if* $\alpha z_\lambda < Z$*.*

Notice from equation (23) that the steady state Z is independent of α. Thus, we can comfortably interpret the condition $\alpha z_\lambda < Z$ as a threshold level for α.

3 The effects of wealth taxation

In this section, we will consider a "tax reform" where we increase the wealth tax gradually and reduce the capital income tax. We will first analyze what happens to aggregate variables and welfare as we increase τ_a from zero. The analysis here applies to any starting level for τ_k and τ_a and is global. In particular, it covers the special case where we start with the capital income tax economy with $\tau_k = \theta > 0$ and $\tau_a = 0$ and then replace the capital income tax with a wealth tax. Next, we will study the optimal combination of capital income and wealth taxes that maximizes total welfare. We abstract from other taxes and transfers to focus on the comparison between capital income and wealth taxes. As mentioned earlier, we maintain the assumption that $\lambda < \overline{\lambda}\,(\tau_a = 0)$ so that the economy features heterogeneous returns before the change in tax policies.

Figure 3: Tax Reform: Switch from Capital Income Tax to Wealth Tax



Note: *The figure plots* $h(x) = (1-\rho\beta(1-\tau_a))\, x^2 - {}^{(z_l+z_\lambda)}/_2\, (1+\rho(1-2\beta(1-\tau_a)))\, x + z_l z_\lambda \rho (1-\beta(1-\tau_a))$ *for two levels of wealth taxes. The steady state productivity corresponds to the larger root of* h*, marked with a circle on the horizontal axis. The red curve corresponds to an increase in wealth taxes* τ_a*.*

3.1 Steady state effects of an increase in wealth taxes

We start by proving a general result describing how the steady state value of Z varies with wealth taxes. The result is independent of budget balancing constraints and states that the value of Z is increasing in the wealth tax rate (τ_a) as long as $\rho > 0$.

Proposition 2. ***(Efficiency Gains from Wealth Taxation)*** *For all* $\tau_a < \overline{\tau}_a$*, a marginal increase in wealth taxes* (τ_a) *increases aggregate productivity* (Z)*,* $\frac{dZ}{d\tau_a} > 0$*, if and only if entrepreneurial productivity is autocorrelated,* $\rho > 0$*.*

As Figure 3 shows, an increase in τ_a shifts the steady state value of Z to the right. Geometrically, this happens only when $\rho > 0$ and the y-intercept of h increases. The values of the parabola are fixed at z_l and z_λ, so the increase of the y-intercept forces the x-intercepts (the roots of h) to shift right. As we discuss below, these shifts reflect the effect of wealth taxes on the after-tax returns, increasing them for high-productivity entrepreneurs and lowering them for low-productivity entrepreneurs. This is the static use-it-or-lose-it effect of wealth taxes. Only when productivity is persistent this increase in the dispersion of returns is translated into higher wealth accumulation by the high-productivity entrepreneurs. As the wealth-share of the high-productivity entrepreneurs rises so does overall productivity Z.

The effect of the reform on after-tax returns is not immediately clear because an increase in wealth taxes also leads to an increase in the effective capital stock Q (see Corollary 4 below), which reduces returns reflecting the decreasing returns to capital in the aggregate. But the returns also respond directly to changes in the tax system which is shifting from τ_k to τ_a. In this way, we can decompose the change in after-tax returns into a general equilibrium and a use-it-or-lose-it component using (22):

$$\frac{dR_l}{d\tau_a} = \underbrace{\left(\frac{z_l}{Z} - 1\right)}_{\text{use-it-lose-it}<0} \underbrace{-\left(\frac{1}{\beta} - (1-\tau_a)\right)\frac{z_l}{Z^2}\frac{dZ}{d\tau_a}}_{\text{G.E. effect}<0} < 0,$$

$$\frac{dR_h}{d\tau_a} = \underbrace{\left(\frac{z_\lambda}{Z} - 1\right)}_{\text{use-it-lose-it}>0} \underbrace{-\left(\frac{1}{\beta} - (1-\tau_a)\right)\frac{z_\lambda}{Z^2}\frac{dZ}{d\tau_a}}_{\text{G.E. effect}<0} > 0.$$

The general equilibrium effect is negative on both types, but the use-it-or-lose-it effect is negative for the low-productivity entrepreneurs and positive for the high-productivity ones. Thus, an increase in wealth taxes unambiguously reduces the returns of low-productivity entrepreneurs, and, from Lemma 2, we know that the use-it-or-lose-it effect dominates for the high-productivity entrepreneurs, so that their after-tax returns increase with the wealth tax, in turn increasing the dispersion of returns. Crucially, the effect of wealth taxes on returns in equilibrium is independent of the assumptions over the government budget. This is because in steady state the level of capital adjusts according to (21) in such a way that returns depend only on overall productivity and wealth taxes (see equation 21). The increase in the dispersion of returns also leads to more wealth concentration as evidence by a higher wealth share of the high-productivity entrepreneurs. Finally, both the arithmetic and the geometric average of returns go down with wealth taxes. We summarize these results next.

Corollary 3. ***(Wealth Shares, Rates of Return, and Wealth Taxes)*** *For all $\tau_a < \overline{\tau}_a$, a marginal increase in wealth taxes (τ_a) increases the share of wealth of high-productivity entrepreneurs $\left(\frac{ds_h}{d\tau_a} > 0\right)$, decreases the after-tax return of low-productivity entrepreneurs $\left(\frac{dR_l}{d\tau_a} < 0\right)$, decreases the after-tax return of high-productivity entrepreneurs $\left(\frac{dR_h}{d\tau_a} > 0\right)$, and decreases the arithmetic and geometric average of returns $\left(\frac{d(R_h+R_l)}{d\tau_a} < 0 \quad \text{and} \quad \frac{d(R_hR_l)}{d\tau_a} < 0\right)$ if and only if $\rho > 0$.*

The implications of the tax reform for the aggregate variables follow directly from Lemma 3 as we summarize below.

Corollary 4. ***(Aggregate Variables after a Tax Reform)*** *For all $\tau_a < \overline{\tau}_a$ and under Assumption 1, effective capital Q, aggregate capital K, aggregate output Y, aggregate wage w, and the wealth of high-productivity entrepreneurs A_h increase after a marginal increase in wealth taxes (τ_a) if and only if $\rho > 0$. The wealth of low-productivity entrepreneurs A_l decreases if and only if $\alpha z_\lambda < Z$.*

Notice that the gains from wealth taxes arise despite the fact that wasteful government expenditure G increases due to Assumption 1. Thus, K, Q, Y, and w could be increased further if G were kept constant in a revenue-neutral fashion. On the other hand, if $\rho < 0$, an increase in wealth taxes not only decreases these variables but also G.

We conclude this section by commenting on the effects of a sustained increase in wealth taxes. As we continue to increase τ_a, Z increases further and will get closer to its upper bound z_h.[6] From this observation, it might be tempting to conclude that we should increase τ_a as much as possible and subsidize capital income. However, maximizing Z (or equivalently the effective capital Q or output Y) does not maximize total welfare. While total income increases as we increase the wealth tax, the dispersion in rates of return also increases (Lemma 3), which works against the welfare of entrepreneurs. In the optimal tax analysis, we will study the optimal combination of τ_k and τ_a that maximizes total welfare.

3.2 Welfare effects of an increase in wealth taxes

We first discuss our welfare measures. Let $\{c_{k,t}(a,i)\}$ and $\{c_{a,t}(a,i)\}$ be consumption paths of an individual of type $i \in \{w,h,l\}$ with the initial wealth a under capital income and wealth tax economies respectively. Note that $i = w$ refers to workers and workers have zero wealth. Define value functions $V_k(a,i)$ and $V_a(a,i)$ similarly. For each individual of type i, we first ask how much they value being dropped from the capital income tax economy with $\tau_a = 0$ and $\tau_k = \theta$ to the economy with a positive wealth tax $\tau_a > 0$ in terms of lifetime consumption. We denote this consumption-equivalent welfare measure as $\mathrm{CE}_1(a,i)$:

$$E\sum_t \beta^{t-1}\log\left(\left(1+\mathrm{CE}_1(a,i)\right)c_{k,t}(a,i)\right) \quad = \quad E\sum_t \beta^{t-1}\log\left(c_{a,t}(a,i)\right). \tag{32}$$

[6] τ_k would adjust in the background to balance the government budget and become a subsidy if τ_a is greater than the tax reform level $\tau_a^{TR} \equiv \frac{\theta(1-\beta)}{\beta(1-\theta)}$.

Then, the welfare gain of an individual of type i satisfies the following expression:

$$\log\left(1+\mathrm{CE}_1\left(a,i\right)\right) \;=\; \left(1-\beta\right)\left(V_a\left(a,i\right)-V_k\left(a,i\right)\right). \tag{33}$$

All terms containing wealth cancel, thus, the welfare gain depends only on the individual's type i. Consequently, we drop wealth "a" from the welfare measure below and write

$$\log\left(1+\mathrm{CE}_{1,i}\right) \;=\; \begin{cases} \log {}^{w_a}\!/_{w_k} & \text{if } i=w \\ \frac{(1-\beta)\log {}^{R_{a,i}}\!/_{R_{k,i}}+\beta(1-p)\left(\log {}^{R_{a,l}}\!/_{R_{k,l}}+\log {}^{R_{a,h}}\!/_{R_{k,h}}\right)}{(1-\beta)(1-\beta(2p-1))} & \text{if } i\in\{l,h\}. \end{cases} \tag{34}$$

Since the $\mathrm{CE}_{1,i}$ measures are independent of asset holdings, the aggregate welfare gain can be computed as the weighted average of welfare gains of each type without worrying about the wealth distribution within each type.[7] Letting n_i be the population share of each type ($n_w \equiv L/(L+2)$ and $n_h = n_l \equiv 1/(L+2)$), the CE_1 utilitarian welfare measure can be written as

$$\log\left(1+\mathrm{CE}_1\right)=\sum_i n_i \log\left(1+\mathrm{CE}_{1,i}\right).$$

We also define the average welfare gain of an entrepreneur ($\mathrm{CE}_1^{\mathrm{e}}$) as

$$\log\left(1+\mathrm{CE}_1^{\mathrm{e}}\right)=\sum_{i\in\{h,l\}}\frac{1}{2}\log\left(1+\mathrm{CE}_{1,i}\right)=\frac{1}{1-\beta}\left(\log {}^{R_{a,l}}\!/_{R_{k,l}}+\log {}^{R_{a,h}}\!/_{R_{k,h}}\right).$$

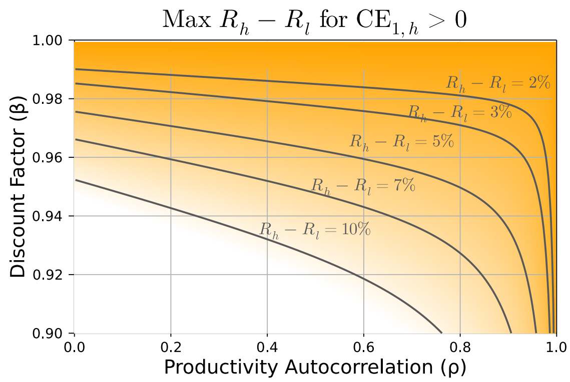
Having defined our welfare measures, we can now use the results of Section 3 to determine the welfare implications of a tax reform following a continuous increase in the wealth tax. Lemma 3 characterizes how the welfare of each type of agent changes after the tax reform.

Proposition 3. ***(Welfare Gain by Agent Type)*** *For all $\tau_a < \overline{\tau}_a$, if Assumption 1 holds and $\rho > 0$, a marginal increase in wealth taxes (τ_a) increases the welfare of workers ($CE_{1,w} > 0$) and decreases the welfare of low-productivity entrepreneurs ($CE_{1,l} < 0$) and the average welfare of entrepreneurs ($CE_1^{\mathrm{e}} < 0$). Furthermore, there exists a upper bound on the dispersion of returns (κ_R) such that an increase in wealth taxes increases the welfare of high-productivity entrepreneurs ($CE_{1,h} > 0$) if and only if $R_h - R_l < \kappa_R$.*

Workers gain from the tax reform since wages increase with the reform. For

[7] It is well understood that there is no stationary distribution of assets in models such as ours, with constant savings rates and no reflecting barrier or resetting mechanisms (Gabaix, 2009). In Section 4, we explore the implications for wealth taxes in a modified version of the model that does allow for a stationary distribution of assets.

Figure 4: Dispersion of Returns and Welfare Gains for High-Productivity Entrepreneurs



Note: *The figure reports the upper bound on the steady state dispersion of returns, $R_h - R_l$, for which $CE_{1,h} > 0$. The upper bound is a function of only ρ and β and is obtained by finding the upper bound on the wealth share of high-productivity entrepreneurs implied by equation (90) and evaluating the returns at that level using the results from Lemma 2. We set the remaining parameters as follows: $z_l = 0$, $z_h = 2$, $\tau_k = 25\%$, $\lambda = 1.32$, and $\alpha = 0.4$.*

entrepreneurs, the effects of the increase in wealth taxes operate through changes in after tax returns. There are two effects. First, higher wealth taxes reduce the returns of low-productivity entrepreneurs and increase those of high-productivity entrepreneurs, this directly reduces the welfare of low-productivity entrepreneurs and increases the welfare of high-productivity entrepreneurs. Second, the (log-) average of returns decreases with wealth taxes (Lemma 3), which decreases entrepreneurs' expectations over future returns, reducing the welfare of both types of entrepreneurs. The net result of these effects is a lower welfare for the low-productivity entrepreneurs and for entrepreneurs as a group.

The welfare gain for the high-productivity entrepreneurs depends on the magnitude of the decrease in average returns. The upper bound on the dispersion of returns (κ_R) ensures that the loss from lower expected returns is low relative to the increase in R_h. As wealth taxes increase returns become more dispersed, in turn decreasing the welfare gain for high-productivity entrepreneurs. The upper bound for the dispersion of returns does not change with wealth taxes, it is in fact only a function of β and p. Figure 4 presents the upper bound on the dispersion of returns for which $\mathrm{CE}_{1,h}$ is positive for different combinations of parameters.

Lemma 3 makes clear the key tradeoff when considering the welfare effects of wealth taxation. Higher taxes increase the welfare of workers by increasing wages through productivity gains and capital accumulation, but they (on average) reduce the welfare of entrepreneurs by hurting low-productivity entrepreneurs and increasing the dispersion of returns. As we show below, the relative magnitudes in this tradeoff are captured by how much wages and returns react to the efficiency gains from wealth taxation. In particular, the welfare gain of workers is proportional to the elasticity of wages (w) with respect to productivity (Z), $\frac{\alpha}{1-\alpha}$, while the welfare loss of entrepreneurs is proportional to the average elasticity of returns (R_l, R_h) with respect to productivity.

The optimal tax combination (τ_a^*, τ_k^*) that maximizes the utilitarian welfare measure CE_1 exactly balances the tradeoff between changes in wages and changes in returns by equating the elasticity of wages with respect to Z with the average elasticity of returns, weighted by population size. Proposition 4 makes this precise:

Proposition 4. ***(Optimal CE*$_1$ *Taxes)*** *If Assumption 1 holds, there exist a unique tax combination (τ_a^*, τ_k^*) that maximizes the utilitarian welfare measure CE$_1$, an interior solution $\tau_a^\star < \overline{\tau}_a$ is the solution to:*

$$n_w \xi_w + \frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_l}+\xi_{R_h}}{2}\right) = 0 \tag{35}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Furthermore, there exist two cutoff values for α which we denote $\underline{\alpha}$ and $\overline{\alpha}$ such that $(\tau_a^\star, \tau_k^\star)$ satisfies the following properties:

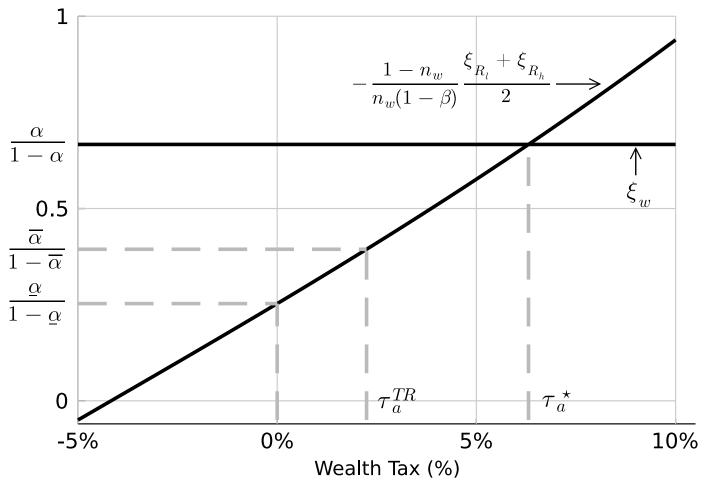
$$\begin{aligned}
\tau_a^\star &\in \left[1-\frac{1}{\beta}, 0\right) \text{ and } \tau_k^\star > \theta && \text{if } \alpha < \underline{\alpha} \\
\tau_a^\star &\in \left[0, \frac{\theta(1-\beta)}{\beta(1-\theta)}\right] \text{ and } \tau_k^\star \in [0,\theta] && \text{if } \underline{\alpha} \le \alpha \le \bar{\alpha} \\
\tau_a^\star &\in \left(\frac{\theta(1-\beta)}{\beta(1-\theta)}, \tau_a^{\max}\right) \text{ and } \tau_k^\star < 0 && \text{if } \alpha > \bar{\alpha}
\end{aligned}$$

where $\underline{\alpha}$ and $\overline{\alpha}$ are the solutions to equation (35) with $\tau_a = 0$ and $\tau_a = \tau^{TR} = \frac{\theta(1-\beta)}{\beta(1-\theta)}$, respectively. Recall from Lemma (3) that $\xi_w = \xi \equiv {}^{\alpha}/_{1-\alpha}$.

Figure 5 illustrates the forces at play. The elasticity of wages with respect to productivity (ξ_w) gives the (percentage) gain in workers' welfare as wealth taxes increase (raising productivity). Recall from Proposition 2 that productivity increases in wealth taxes regardless of the response of other taxes or aggregates. In the economy we model, the elasticity of wages with respect to productivity is constant ($\xi_w = {}^{\alpha}/_{1-\alpha}$) following the functional form of the production function in equation (1).[8] On the other hand, the

[8] More generally, the elasticity of wages with respect to productivity is weakly decreasing. Consider a

Figure 5: Optimal Wealth Tax



Note: *The figure shows the conditions satisfied by the optimal wealth tax, defined as the tax that maximizes* CE_1*. The horizontal line is the elasticity of wages with respect to productivity* (ξ_w)*. The increasing line is proportional to the average elasticity of returns with respect to productivity* (ξ_R)*.* $\tau_a^\star$ *denotes the optimal wealth tax.* $\tau_a^{TR} = \theta(1-\beta)/\beta(1-\theta)$ *denotes the tax reform tax, the level at which* $\tau_k = 0$*. The remaining parameters are as follows:* $\beta = 0.96$*,* $p = 0.9$*,* $z_l = 1/2$*,* $z_h = 3/2$*,* $\theta = 25\%$*, and* $\lambda = 1.2$*.*

(negative) average elasticity of returns is increasing, reflecting the widening gap between low- and high-productivity entrepreneurs as wealth taxes increase. The intersection of the two lines marks the optimal wealth tax.

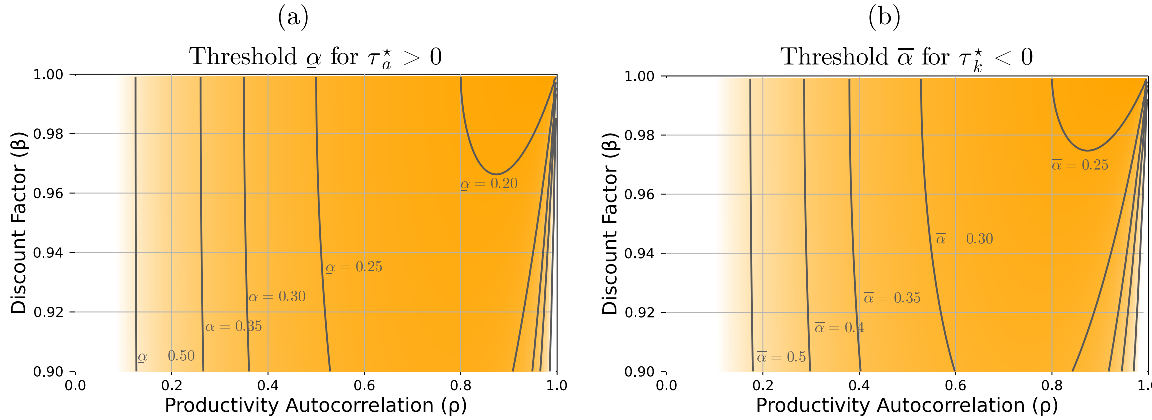
Figure 5 also helps to clarify the role of the thresholds $\underline{\alpha}$ and $\overline{\alpha}$.[9] In effect, $\underline{\alpha}$ marks the level of ξ_w at which $\tau_a = 0$ would be optimal, any $\alpha > \underline{\alpha}$ increases the gains for workers from wealth taxation and implies a higher optimal wealth tax. With higher α there are less diminishing marginal returns in $Q = ZK$, and a higher scope for wages to rise as Q increases with the wealth tax. The value of $\overline{\alpha}$ is similarly defined by the level of ξ_w at which $\tau_a = \tau_a^{TR} = \theta(1-\beta)/\beta(1-\theta)$ is optimal. At that level wealth taxes finance all government spending and $\tau_k = 0$. Consequently, any $\alpha > \overline{\alpha}$ implies that the optimal tax combination is one of wealth taxes and capital income subsidies. Finally, the upper bound on the wealth tax ($\tau_a^{\max}$) ensures that R_l remains positive.

To give an idea of the level of the thresholds $\underline{\alpha}$ and $\overline{\alpha}$, Figure 6 presents values for different combinations of β and p. We keep the dispersion of productivities as in Figure 2 and set $\lambda = 1.32$ and $\theta = 0.25$, we also set $L = 18$ so that 10% of households are

cosntant-returns-to-scale and concave technology $f(ZK, L)$, then $\xi_w \equiv Z\frac{df_2\left(1, \frac{L}{ZK(Z)}\right)}{dZ} = \frac{-L}{wQ} f_{22}\left(1, \frac{L}{Q}\right)\xi_Q$.

[9] It might seem strange that we have endogenous variables s_h and Z in these thresholds. However, s_h and Z are determined through equation (130) and are independent of α, so they can be used to define the threshold.

Figure 6: Conditions for Welfare Gains from Wealth Taxation



Note: *Figure 6a reports the value of $\underline{\alpha}$ found in Proposition 4 for combinations of the autocorrelation of productivity (ρ) and the discount factor (β). Positive wealth taxes induce welfare gains (as measured by CE_1) if $\alpha \geq \underline{\alpha}$. Figure 6b reports the value of $\overline{\alpha}$ found in Proposition 4. Positive wealth taxes and capital income subsidies induce welfare gains (as measured by CE_1) if $\alpha \geq \overline{\alpha}$. In both figures we set the remaining parameters as follows: $z_l = 0$, $z_h = 2$, $\theta = 25\%$, and $\lambda = 1.32$.*

entrepreneurs. The levels of $\underline{\alpha}$ and $\overline{\alpha}$ are both lower than 0.3 if productivity is sufficiently persistent ($\rho > 0.6$), although, interestingly, the thresholds aren't monotone and they go up as persistence approaches 1. The non-monotonicity arises due to two opposing forces as ρ increases. First, there is less misallocation of capital since wealth is more concentrated in the hands of more productive entrepreneurs. Thus, there is less scope for improvement from wealth taxes. On the other hand, the effect of the use-it-or-lose-it mechanism on misallocation gets stronger as ρ increases.

We can also show that $\bar{\alpha}$ increases with θ, the size of the government's budget. However, the optimal τ_a is independent of θ. Thus, the government is more likely to use a combination of positive capital income and wealth taxes for larger government budget, captured by higher θ.

The optimal level of wealth taxes exhibits the same pattern of increasing in persistence up until ρ approaches 1. Welfare gains are close to zero if persistence is relatively low ($\rho < 0.6$), but grow rapidly as ρ increases with the highest levels associated with high levels of β. We report the optimal tax levels and welfare gains for combinations of β and ρ in Figure D.4 of Appendix D.

Finally, we note that it is possible to solve for all the aggregates of the model explicitly in terms of parameters if we assume $z_l = 0$. In this case we can also solve for the thresholds $\underline{\alpha}$ and $\overline{\alpha}$. Under this assumption ($z_l = 0$), the steady state Z is given as $Z = \frac{z_\lambda}{2} \frac{1+\rho(1-2\beta(1-\tau_a))}{1-\rho\beta(1-\tau_a)}$, and after-tax rates of return are $R_l = 1 - \tau_a$ and $R_h = \frac{2-(\rho+1)\beta(1-\tau_a)}{\beta(1+\rho(1-2\beta(1-\tau_a)))}$. We can use the results in Lemma 3 to obtain closed form expressions of the remaining aggregate variables in terms of parameters and taxes, which we present in the next Corollary:

Corollary 5. ***(α-Thresholds)*** *If $z_l = 0$, the thresholds $\underline{\alpha}$ and $\bar{\alpha}$ are explicitly given by* $\frac{\underline{\alpha}}{1-\underline{\alpha}} = \frac{1+\rho}{L\rho(1-\rho)} \frac{(1-\rho\beta)^2}{\beta\left(1-\beta\frac{1+\rho}{2}\right)}$ *and* $\frac{\bar{\alpha}}{1-\bar{\alpha}} = \frac{1+\rho}{L\rho(1-\rho)} \frac{(1-\theta-\rho(\beta-\theta))^2}{(1-\theta)(\beta-\theta)\left((1-\theta)-\frac{1+\rho}{2}(\beta-\theta)\right)}$.

3.2.1 Taking wealth accumulation into account

The welfare of entrepreneurs depends on three different but intertwined components: The level of their after-tax returns (R_i), the (log-)average returns, and their asset holdings of the entrepreneurs (see equation 8). However, the CE_1 measures above only capture the first two components, ignoring the effects of the increase in aggregate capital K and the share of wealth held by the high-productivity entrepreneurs s_h brought about by the tax reform. This asymmetry can generate welfare losses from wealth taxation to even high-productivity entrepreneurs as we proved in Lemma 3.

As an alternative to the CE_1 used above, we also consider the welfare gain of a stand-in representative agent of each type. This takes into account the change in average wealth for the entrepreneurs across tax systems. This does not affect workers because they hold no assets. For entrepreneurs, we compare the values assigned by a type-i entrepreneur of being in the capital income or wealth tax economy while holding the average type-i wealth level in that economy. We denote this welfare measure as $\text{CE}_{2,i}$:

$$\log\left(1+\text{CE}_{2,i}\right) = (1-\beta)\left(V_a\left(A_{i,a},i\right) - V_k\left(A_{i,k},i\right)\right) = \log\left(1+\text{CE}_{1,i}\right) + \log\left({}^{A_{a,i}}/_{A_{k,i}}\right). \quad (36)$$

Corollary 4 implies A_l goes down with the tax reform if α is not too high. Thus, we expect that $\text{CE}_{2,l} < \text{CE}_{1,l}$. On the other hand, as we show in Lemma 4, the high types unambiguously benefit from wealth taxes once the increase asset holdings is taken into account.

Lemma 4. ***(Welfare Gains by Agent Type with Asset Accumulation)*** *For all $\tau_a < \overline{\tau}_a$, if Assumption 1 holds and $\rho > 0$, a marginal increase in wealth taxes (τ_a) increases the welfare of high-productivity entrepreneurs ($CE_{2,h} > 0$). The welfare of low-productivity entrepreneurs decreases ($CE_{2,l} < 0$) if $\alpha z_\lambda < Z$.*

We can also ask each entrepreneur behind the veil of ignorance how much she values being born in the wealth tax economy with the average wealth of the wealth tax economy relative to being born in the capital income tax economy with the average wealth of the capital income tax economy. If the individual is born as type i, her welfare gain would be

$$\log\left(1+\widetilde{\mathrm{CE}}_{2,i}\right)=(1-\beta)\left(V_a\left(K_a,i\right)-V_k\left(K_k,i\right)\right)=\log\left(1+\mathrm{CE}_{1,i}\right)+\log\left({}^{K_a}/_{K_k}\right), \quad (37)$$

and the aggregate welfare is

$$\log\left(1+\widetilde{\mathrm{CE}}_{2}\right)=\sum_i n_i\log\left(1+\widetilde{\mathrm{CE}}_{2,i}\right)=\log\left(1+\mathrm{CE}_{1,i}\right)+\log\left({}^{K_a}/_{K_k}\right). \quad (38)$$

The increase in K with the wealth tax contributes positively to $\widetilde{\mathrm{CE}}_{2,i}$ measure for entrepreneurs. Thus, the aggregate welfare increases more under this measure compared to CE_1 and the α thresholds needed for higher wealth taxes will lower. Interestingly, this does not alter the nature of the key tradeoff we described in Proposition 4. Optimal taxes are still trading off the efficiency gains from wealth taxation with the loses to entrepreneurs from higher return dispersion. In fact, we know from Lemma 3 that $\xi_w = \xi_K = \xi$, so that incorporating the gains from higher capital accumulation acts by re-weighting the gains from wealth taxes. In proposition 5 we present the optimal tax outcomes when the government's objective to maximize $\widetilde{\mathrm{CE}}_2$.

Proposition 5. ***(Optimal $\widetilde{CE}_2$ Taxes)*** *If Assumption 1 holds, there exist a unique tax combination $\left(\tau^\star_{a,2},\tau^\star_{k,2}\right)$ that maximizes the utilitarian welfare measure $\widetilde{CE}_2$, an interior solution $\tau^\star_{a,2}<\overline{\tau}_a$ is the solution to:*

$$n_w\xi_w+(1-n_w)\,\xi_K=\xi=-\frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_l}+\xi_{R_h}}{2}\right) \quad (39)$$

where $\xi_x\equiv\frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Furthermore, there exist two cutoff values for α which we denote $\underline{\alpha}_2$ and $\overline{\alpha}_2$ such that $\left(\tau^\star_{a,2},\tau^\star_{k,2}\right)$ satisfies the following properties:

$$\begin{aligned}
&\tau^\star_{a,2}\in\left[1-\frac{1}{\beta},0\right) \text{ and } \tau^\star_{k,2}>\theta && \text{if } \alpha<\underline{\alpha}_2\\
&\tau^\star_{a,2}\in\left[0,\frac{\theta(1-\beta)}{\beta(1-\theta)}\right] \text{ and } \tau^\star_{k,2}\in[0,\theta] && \text{if } \underline{\alpha}_2\le\alpha\le\bar{\alpha}_2\\
&\tau^\star_{a,2}\in\left(\frac{\theta(1-\beta)}{\beta(1-\theta)},\tau^{\max}_{a,2}\right) \text{ and } \tau^\star_{k,2}<0 && \text{if } \alpha>\bar{\alpha}_2
\end{aligned}$$

where $\underline{\alpha}_2$ and $\overline{\alpha}_2$ are the solutions to equation (39) with $\tau_a = 0$ and $\tau_a = \frac{\theta(1-\beta)}{\beta(1-\theta)}$, respectively. Recall from Lemma (3) that $\xi = {}^{\alpha}/_{1-\alpha}$.

Taking into account the role of capital accumulation results in a higher optimal tax level, and lower thresholds $\underline{\alpha}$ and $\overline{\alpha}$:

Corollary 6. ***(Comparison of CE$_1$ and CE$_2$ Taxes)*** *Optimal wealth taxes are higher when taking the wealth accumulation into account $\left(\tau^{\star}_{a,2} > \tau^{\star}_{a}\right)$. Moreover, the α-thresholds are lower $\underline{\alpha}_2 < \underline{\alpha}$ and $\overline{\alpha}_2 < \overline{\alpha}$.*

4 Extensions

4.1 Corporate sector

In the model of Section 2 all production is carried out by entrepreneurs who face collateral constraints, this ignores the role of the corporate sector. Incorporating the corporate sector into the model also provides entrepreneurs with an alternative use for their assets, as they can invest in the corporate sector rather than use their assets in their own productive endeavors. However, we show that incorporating the corporate sector does not substantively affect any of our results. In the most relevant case where the corporate sector productivity is in between the productivities of the high- and low-productivity entrepreneurs, the corporate sector takes up the production role of low-productivity entrepreneurs, who now invest in their assets in the corporate sector instead of using their own technology. All other results remain unchanged with the exception that the corporate sector's productivity takes up the role of z_l.

We consider a model like that in Section 2 where there is also a corporate sector that produces the same final good as the entrepreneurs using a constant-returns-to-scale technology:

$$Y_c = (z_c K_c)^{\alpha} L_c^{1-\alpha}. \tag{40}$$

Unlike the entrepreneurs, the corporate sector faces no collateral constraints. This imposes a lower bound on the equilibrium rental rate of capital r which cannot be lower than the marginal return on capital in the corporate sector:[10]

$$r \geq \alpha z_c \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}.$$

[10]Otherwise the problem of the corporate sector would be ill-defined and the corporate demand for capital would be infinity. See Appendix C.1 for details.

If $z_c < z_l$ the corporate sector does not operate ($Y_c = 0$) and the economy works as in Section 2. Because z_c is low relative to the productivity of entrepreneurs, low-productivity entrepreneurs can attain a higher rate of return by operating their own technology rather than lending assets to the corporate sector.

If $z_c > z_h$, only the corporate sector operates in equilibrium using the assets of all entrepreneurs, who receive a higher return this way than operating their own technologies. This equilibrium is efficient. We discuss the knife-edge cases of $z_c = z_h$ and $z_c = z_l$ in Appendix C.1.

The more interesting case arises when $z_l < z_c < z_h$. In this scenario both the corporate sector and the high-productivity entrepreneurs operate in equilibrium, while the low-productivity entrepreneurs do not produce and instead lend all of their funds to the corporate sector. Even though the corporate sector is operating in equilibrium, there are no real changes in the aggregates of the economy. As in Section 2, we focus on the equilibrium with heterogeneous return where the high-productivity entrepreneurs are constrained in their demand for capital and demand $K_h = \lambda A_h$, but now the remaining capital is used by the corporate sector rather than by the low-productivity entrepreneurs.[11] The corporate sector makes zero profits in equilibrium and $r = \alpha z_c \left({}^{1-\alpha}/_{w}\right)^{1-\alpha/\alpha}$, so that z_c takes the place of z_l in determining the interest rate in the economy. Total output is still given by

$$Y = (ZK)^{\alpha} L^{1-\alpha},$$

but the wealth-weighted productivity of capital is now $Z = s_h z_\lambda + s_l z_c$, where $z_\lambda = z_h + (\lambda - 1)(z_h - z_c)$. The change to z_c also affects the after-tax returns of the entrepreneurs which now are:

$$R_l = (1 - \tau_a) + (1 - \tau_k)\, \alpha \left({}^{ZK}/_{L}\right)^{\alpha-1} z_c$$
$$R_h = (1 - \tau_a) + (1 - \tau_k)\, \alpha \left({}^{ZK}/_{L}\right)^{\alpha-1} z_\lambda.$$

These changes do not affect the derivation of any of the results above, in particular Propositions 1, 2, and 4 characterizing the equilibrium level of productivity, the efficiency gains from wealth taxation and the optimal tax schedule. The main consequence of the change from z_l to z_c is that the relevant dispersion of productivities is now z_h versus z_c,

[11] In this equilibrium, $K_l = 0$ and $K = A_h + A_l = K_h + K_c$. The return of capital for the high-productivity entrepreneurs is $\alpha z_h \left({}^{1-\alpha}/_{w}\right)^{1-\alpha/\alpha} > r$.

which is lower than it was in Section 2 (recall that $z_c > z_l$). This reduces the misallocation and thus the scope for efficiency gains. Lemma 5 in Appendix C.1 formalizes these results.

4.2 Entrepreneurial effort

In the model of Section 2 entrepreneurs' productivity is exogenously determined, this ignores the role of entrepreneurial effort in shaping the productivity of private enterprises, as well as the role of the tax system in affecting the incentives to entrepreneurs to exert more effort in the management of their firms. In fact, both capital income and wealth taxes can affect the incentives of entrepreneurs to exert effort and increase the productivity of their firms. Capital income taxes reduce the portion of profits that entrepreneurs can retain, lowering the benefits to additional effort by the entrepreneurs. Wealth taxes can lead to lower wealth accumulation, which reduces the amount of capital an entrepreneur can use in their firm, in turn reducing the marginal product of any additional effort the entrepreneur would exert.

We introduce entrepreneurial effort into the model developed in Section 2 in a tractable manner that allows us to identify the core implications for the effects of wealth taxes. We capture the effect of effort as modifying the production function of entrepreneurs to:

$$y = (zk)^{\alpha} e^{\gamma} n^{1-\alpha-\gamma}. \tag{41}$$

where $0 \leq \gamma < 1-\alpha$. Exerting effort has a utility cost that we capture in by modifying the utility function to

$$u(c,e) = \log(c - h(e)),$$

where $h(e) = \psi e$ with $\psi > 0$.[12] Tractability depends on preserving the constant-returns-to-scale of the production function and abstracting from income effects in the effort choice as in Greenwood et al. (1988).[13] These two elements together lets us solve the model in closed form. The solution inherits the same properties of our baseline model after a suitable change of variables. Letting $\hat{c} = c - h(e)$ and re-defining profits to include the cost of effort

[12] In general we can let effort affect production according to an increasing function $g(e)$ and we only require that the ratio $h'(e)/g'(e)$ is constant. See Appendix C.2.

[13] Abstracting from the income effect on the entrepreneur's effort choice potentially leads to an overstatement of the response of effort to wealth taxes. Wealth taxes increase the returns of entrepreneurs and incentivize effort, but richer entrepreneurs may also want to exert less effort in the presence of income effects.

as:

$$\hat{\pi}(z,k) = \max_{n,e} y - wn - rk - \frac{1}{1-\tau_k} h(e) \tag{42}$$

Crucially, capital income taxes have a direct effect on effort. Taxes are paid on profits in such a way that labor and capital rental costs can be deducted, while effort costs are paid privately by the entrepreneur and are not deductible from taxes. Because of this the effective cost per unit of effort is $\psi/1-\tau_k$.

The solution to the entrepreneur's problem is linear in capital, as in Section 2, and allows us to express both profits and effort as $\hat{\pi}(z,k) = \pi^\star(z)\,k$ and $e(z,k) = \varepsilon(z)\,k$ for suitable functions $\pi^\star(z)$ and $\epsilon(z)$. Finally, we can define an auxiliary variable capturing the after-tax returns net of effort cost:

$$\hat{R}(z) \equiv (1-\tau_a) + (1-\tau_k)(r + \pi^\star(z)\,\lambda). \tag{43}$$

The variable $\hat{R}(z)$ plays an important role because it affects the savings choice of entrepreneurs which is now $a' = \beta\hat{R}(z)\,a$. The details of these derivations are presented in Appendix C.2.

The linearity of the solution allows us to aggregate and obtain closed form expressions for equilibrium quantities as a function of aggregate capital, K, and productivity, Z, paralleling the results of Lemma 1 in Section 2. The main difference is of course the introduction of effort. Aggregate effort is:

$$E = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{1}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}}. \tag{44}$$

There are two different, but related, forces shaping the total amount of entrepreneurial effort in the economy. First, effort is increasing in capital, K, and productivity, Z. At the same time, effort is disincentivized by capital income taxes, which reduce the after-tax marginal product of effort effectively making effort more costly. Notice that wealth taxes do not directly affect the effort choice because they do not affect the fraction of profits retained by the entrepreneur. This second force also affects aggregate output and wages through effort, with capital income taxes directly reducing Y and w:

$$Y = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}}, \tag{45}$$

$$w = (1-\alpha-\gamma)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{ZK}{L}\right)^{\frac{\alpha}{1-\gamma}}. \tag{46}$$

We present a formal statement of these result in Lemma 6 of Appendix C.2.

Despite these differences, the steady state behavior of the wealth-weighted productivity of capital Z remains unchanged. The relationship between productivity, taxes and steady state capital changes, but the steady state relationship between productivity and the after-tax return net of effort costs is the same as the relationship between productivity and after-tax returns in Section 2.3:

$$\hat{R}(z) = \begin{cases} (1-\tau_a) + \left(\frac{1}{\beta} - (1-\tau_a)\right)\frac{z_l}{Z} & \text{if } z = z_l \\ (1-\tau_a) + \left(\frac{1}{\beta} - (1-\tau_a)\right)\frac{z_\lambda}{Z} & \text{if } z = z_h \end{cases} \tag{47}$$

As a consequence of this, equation (23) that determines the steady state level of Z remains unchanged. The distortive effects of capital income income taxes affect effort and capital accumulation, but have no effect on steady state returns (once the effort cost are accounted for). This is precisely because the steady state level of capital adjusts so that its marginal product (that includes the new capital income tax term) is equal to $\frac{1}{\beta} - (1-\tau_a)$ as in our baseline model:

$$\frac{1}{\beta} - (1-\tau_a) = (1-\tau_k)\,\alpha\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} Z^{\frac{\alpha}{1-\gamma}} \left(\frac{L}{K}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}. \tag{48}$$

These results lets us conclude that the main results of our baseline model regarding the existence of the steady state and the efficiency gains from wealth taxation remain unchanged:

Proposition 6. *Propositions 1 and 2 apply to this economy, so that a steady state equilibrium with heterogeneous returns exists if and only if $\lambda < \overline{\lambda}$, and a marginal increase in wealth taxes in such an equilibrium increases productivity Z if and only if $\rho > 0$.*

Nevertheless, introducing an effort choice for entrepreneurs does change the response of aggregates to changes in wealth taxes and the choice of optimal taxes. As wealth taxes increase productivity rises as before, increasing capital, output, and wages as described in Lemma 3, but higher wealth taxes also reduce the level of capital income taxes (see equation 24). Lower capital income taxes incentivize entrepreneurial effort and trough it increase aggregate output, capital, and wages as equations (44), (45), (46) and (48) make clear.

Lemma 5. *For all $\tau_a < \overline{\tau}_a$, a marginal increase in wealth taxes (τ_a) increases aggregate entrepreneurial effort (E) if $\rho > 0$. The aggregate capital stock, output, and wages also increase: $\frac{dE}{d\tau_a}, \frac{dK}{d\tau_a}, \frac{dY}{d\tau_a}, \frac{dw}{d\tau_a} > 0$.*

The reduction of distortions on effort caused by the increase in wealth taxes adds an additional motive in favor of wealth taxes which impacts the optimal tax choice. Just as in proposition 4 the optimal tax choice balances the gains to workers from a higher wage with the reduction in average after tax returns (now net of effort costs). The response of the after tax returns to taxes is not affected by the effort, as implied by equation (47), but the increase in wages is now augmented via an increase in entrepreneurial effort. Because of this, the optimal the optimal tax combination now involves a higher wealth tax and lower capital income taxes:[14]

Proposition 7. *The optimal wealth tax with entrepreneurial effort is higher than the optimal tax found in Proposition 4. Moreover, if the optimal wealth tax is interior ($\tau_a^\star < \overline{\tau}_a$) it satisfies:*

$$n_w \frac{\gamma}{1-\alpha-\gamma}\left(\frac{\frac{\beta\tau_a}{1-\beta(1-\tau_a)}}{\frac{d\log Z}{d\log\tau_a}} + \frac{\alpha}{1-\alpha}\right) = -\left(n_w \frac{\alpha}{1-\alpha} + \frac{1-n_w}{2}\left(\xi_{\hat{R}_l} + \xi_{\hat{R}_h}\right)\right)$$

where $\xi_{\hat{R}_l}$ and $\xi_{\hat{R}_h}$ are equivalent to the elasticities of after-tax returns with respect to productivity in the model without entrepreneurial effort (Section 2).

4.3 Excess returns

In this section, we consider the possibility that returns do not necessarily capture the productivity of individual entrepreneurs and some group of agents receive higher (or lower) return on their assets than their marginal products. We model these excess returns by introducing return wedges: an entrepreneur who earns more (less) than their marginal

[14] The same forces lead to higher wealth taxes when wealth accumulation is taken into account because lowering capital income taxes also increases wealth accumulation.

product is assumed to face a positive (negative) wedge. These wedges allocate resources across groups of individuals and we assume that the net allocation is equal to zero. Thus, it is as if a group of agents are subsidized at the expense of another group and overall these wedges are zero-sum.

To be precise, we consider the case where high (low)-productivity entrepreneurs face a wedge ω_h (ω_l) on their returns, so that the after-tax return becomes

$$R_l = (1-\tau_a) + (1-\tau_k)(1+\omega_l)\,\alpha\left(ZK/L\right)^{\alpha-1} z_l \tag{49}$$

$$R_h = (1-\tau_a) + (1-\tau_k)(1+\omega_h)\,\alpha\left(ZK/L\right)^{\alpha-1} z_\lambda. \tag{50}$$

Given the zero-sum condition is $\omega_l z_l A_l + \omega_h z_\lambda A_h = 0$, the law of motion for aggregate capital is the same as in our benchmark model and hence the steady state condition for aggregate capital is still given by equation (21). In contrast, the return wedges directly affect the steady state condition for Z. After incorporating the zero-sum condition on the wedges, we express the steady state condition for Z in terms of ω_h as

$$(1-\rho\beta(1-\tau_a))Z^2 - \frac{z_\lambda + z_l}{2}(1+\rho(1-2\beta(1-\tau_a)))Z + z_l z_\lambda \rho(1-\beta(1-\tau_a))\left(1+\omega_h-\omega_h\frac{Z}{z_l}\right) = 0. \tag{51}$$

The wedges impose new conditions for the existence of a steady state equilibrium with heterogeneous returns by modifying the threshold level of λ and adding new conditions on the level wealth taxes. We present these conditions on Proposition 8 in Appendix C.3.

Relative to our benchmark case (equivalent to $\omega_h = 0$), an increase in ω_h makes the bound on λ more stringent, $d\bar{\lambda}/d\omega_h < 0$, as $\omega_h > 0$ increases the effective return of high-productivity entrepreneurs. A decrease of ω_h does the opposite. The additional conditions of Proposition 8 on wealth taxes put an upper (lower) limit on τ_a when ω_h is positive (negative). The upper limit is positive iff $\omega_h < \frac{1-\rho}{2\rho(1-\beta)}$, which is always true if $p \leq \beta$ since $\omega_h < 1$. The lower limit is always negative as long as $\rho > 0$.

The introduction of return wages has a more significant effect on the efficiency gains from wealth taxation. There are now two relevant cases. If productivity is persistent, $\rho > 0$ as in our benchmark case, the steady state Z increases with wealth taxes if and only if the after-tax return of high productive entrepreneurs are higher than the after-tax returns of the low productive entrepreneurs, $R_h > R_l$. This is the case as the subsidies to low-productivity entrepreneurs are not too large, effectively imposing a lower bound on ω_h. Intuitively, wealth taxes benefit the agents with higher returns, regardless of their productivity, so that wealth taxes would actually reduce aggregate productivity if high-return individuals are actually unproductive.

The second case arises if productivity is negatively auto-correlated, $\rho < 0$. In this case the low-productivity entrepreneurs today are likely to become productive in the future. Because of this, an increase in wealth taxes increases aggregate productivity if and only if the low-productivity entrepreneurs have higher returns than high-productivity entrepreneurs, $R_l < R_h$. This is exactly the opposite condition as in the first case, precisely because wealth taxes further increase the returns of currently unproductive agents, in turn increasing the wealth share of high-productivity agents in the future. Of course, if there is no underlying difference between groups (i.e., $z_l = z_h$) then the wedges are only redistributing and there is no change in aggregate efficiency.

The following proposition formalizes the analysis. We abuse notation by referring to the modified upper bound on taxes as $\overline{\tau}_a$ the same as before, even though it now depends on the return wedges ω_h and ω_l.

Proposition 8. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a$, a marginal increase in wealth taxes (τ_a) increases aggregate productivity (Z), $\frac{dZ}{d\tau_a} > 0$, ¡if entrepreneurial productivity is autocorrelated, $\rho > 0$, and $R_h > R_l$,or if entrepreneurial productivity is negatively autocorrelated, $\rho < 1$, and $R_h < R$.*

Corollary 7. *For all $\tau_a < \overline{\tau}_a$, the steady state returns satisfy $R_h > R_l$ if and only if :*

$$\omega_h > \underline{\omega}_h = -\frac{1}{2}\left(\frac{1-\rho}{3+\rho}\right)\left(\frac{z_\lambda - z_l}{z_\lambda}\right).$$

4.4 Markups

In this Section, we consider uniform markups charged by firms to consumers. We assume that all entrepreneurial firms charge a uniform and exogenous markup $\mu > 0$. The markups are an additional cost to both workers and entrepreneurs when consuming. This form of uniform markups does not change any of our main results, but it affects the welfare of entrepreneurs who now face an additional cost when purchasing goods, but also have higher average wealth because of the additional revenue generated by markups.

Adding markups changes the problem of the entrepreneurs in two ways. First, markups modify the solution to their production problem by replacing z_i for $z_i^\mu \equiv (1+\mu)^{\frac{1}{\alpha}} z_i$. This change does not affect the formulas in Lemma 1 or the law of motion of assets, but it does change the interpretation of $Z^{\mu} = s_h z_{\lambda}^{\mu} + (1 - s_h)\, z_l^{\mu}$ to be revenue TFP instead of quantity TFP, in fact $Z^{\mu} = (1+\mu)^{\frac{1}{\alpha}}\, Z$. Second, markups enter as an additional cost in the consumption-saving problem. We can easily verify that the savings choice of the consumer is still given by (7), but that the value function has an additional term capturing the markups costs:

$$V_i\,(a) = -\frac{\log\,(1+\mu)}{1-\beta} + m_i^{\mu} + \frac{1}{1-\beta}\log\,(a)\,,$$

where the constant term m_i is as in Section 2.1.

The value function of workers is also modified and includes the same markup cost as the entrepreneurs' value function:

$$V_w = \frac{\log w^{\mu}}{1-\beta} - \frac{\log\,(1+\mu)}{1-\beta}.$$

However, the value function is not affected in equilibrium because equilibrium wages adjust by a factor of $1+\mu$ due to the increase in labor demand in response to the markup's revenue, $w^{\mu} = (1-\alpha)\left({}^{Z^{\mu}K}/_{L}\right)^{\alpha} = (1+\mu)\, w$.

Because markups do not change the evolution of aggregates, it is still the case that a marginal increase of wealth taxes increase (revenue) TFP, but quantity TFP also increases. This follows from the effect of the increase of wealth taxes on wealth concentration among the high-productivity entrepreneurs. It is still the case that s_h increases and so TFP increases following the increase in wealth taxes. The main result regarding the characterization of optimal taxes (proposition 4) also stands. Moreover, the steady state value of returns is left unchanged. Abusing notation slightly we can write the steady returns of entrepreneurs as:

$$R_i = (1-\tau_a) + \left(\frac{1}{\beta} - (1-\tau_a)\right)\frac{z_i^{\mu}}{Z^{\mu}} = (1-\tau_a) + \left(\frac{1}{\beta} - (1-\tau_a)\right)\frac{z_i}{Z}, \tag{52}$$

which is the same expression we had in equation (22). The reason for this is that the level of capital adjusts in steady state to maintain a constant rate of return that depends only on the discount factor β and wealth taxes as in equation (21):

$$(1+\mu)\,(1-\tau_k)\,\alpha Z^{\alpha}\left({}^{K}/_{L}\right)^{\alpha-1} - \tau_a = \frac{1}{\beta} - 1. \tag{53}$$

Equation (53) also implies that the steady state level of capital is higher under markups.

This comes as no surprise because entrepreneurs' returns at any given level of capital are now higher thanks to the markups.

4.5 Stationary wealth distribution

The model presented in Section 2 as well as the extensions presented above do not have a stationary wealth distribution. Here we consider an alternative version of the model in which entrepreneurs have a permanent productivity type but are subject to mortality risk. In particular assume that entrepreneurs die with a constant probability $1-\delta$, upon death they are replaced by a new entrepreneur with initial assets $\overline{a}$ and whose productivity is z_i ($i \in \{h, l\}$) with probability $1/2$. The value of $\overline{a}$ is determined endogenously in equilibrium as the average bequest in the economy (which coincides with the average wealth).

With respect to the main model of Section 2, this perpetual-youth model loses the variation in productivity for individual entrepreneurs. In exchange, this alternative version of the model exhibits a stationary wealth distribution that allows to better study how changes in taxes affect wealth inequality and welfare. An increase in wealth taxes makes the distribution of wealth more unequal, shifting weight towards both the top and the bottom of the distribution. This is a direct consequence of the higher dispersion in after-tax returns. In terms of welfare, all entrepreneurs benefit from the wealth accumulation that follows an increase in wealth taxes, but they are also more sensitive to changes in returns which affect their savings and compound to higher or lower wealth as they age amplifying the positive and negative effects of wealth taxation on entrepreneurs.

The solution of the perpetual-youth model parallels that of our baseline model, with entrepreneurs saving a constant fraction of their income: $a' = \beta\delta R(z)\, a$.[15] This saving behavior along with the death-and-birth dynamics of the model lead to the following evolution equations for the aggregate wealth of each entrepreneurial type:

$$A_i' = \beta\delta^2 R_i A_i + (1-\delta)\,\overline{a}, \tag{54}$$

for $i \in \{l, h\}$. Equation (54) determines the steady state level of wealth of each entrepreneurial type along with the assumption that all $\overline{a}$ is given by the average wealth (capital in the economy): $\overline{a} \equiv K/2 = (A_l + A_h)/2$. These same equations determine the steady state level of capital and productivity, which again parallel the solution found in Section 2.3. The steady state level of capital is such that the (after-tax) marginal product of

[15] We relegate most details to Appendix C.4.

capital is equal to $\frac{1}{\beta\delta} - (1-\tau_a)$, and the steady state productivity is the solution to a quadratic equation not unlike equation (23).

We show in Appendix C.4 that the steady state equilibrium of the economy features heterogenous returns ($R_h > R_l$) if and only if the collateral constraint of entrepreneurs is sufficiently tight: $\lambda < \lambda_p^{\star}$. Moreover, low-productivity entrepreneurs dissave while high-productivity entrepreneurs save in equilibrium: $\beta\delta R_l < 1 < \beta\delta R_h < 1/\delta$. See Proposition 10.

We also show that the mechanism behind the efficiency gains from wealth taxation remains active. In fact, an increase in wealth taxes always leads to an increase in productivity in this economy because individual productivity is permanent (within a generation). We formalize this result in the following proposition:

Proposition 9. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a^p$, a marginal increase in wealth taxes (τ_a) increases productivity , $\frac{dZ}{d\tau_a} > 0$.*

The response of aggregate variables to changes in equilibrium Z (and hence to τ_a) follow the same patterns as in Section 3. In Appendix C.4.5 we present the equivalent results to Lemmas 3 and 3 describing the response of all aggregate variables in steady state.

We now derive the stationary distribution of assets. Note that all entrepreneurs are born with the same level of wealth $\overline{a}$ and then save at a constant rate during their lifetimes. In particular, high-types save at a (gross) rate $\beta\delta R_h > 1$ and low-types dissave at a (gross) rate $\beta\delta R_l < 1$. So, in the stationary equilibrium the wealth distribution of high-types has support in the interval $[\overline{a}, \infty)$ and the distribution of low-types in the interval $(0, \overline{a}]$. Moreover, the distribution of wealth is discrete, with endogenous mass points at $\left\{\overline{a}, \beta\delta R_h \overline{a}, (\beta\delta R_h)^2 \overline{a}, \ldots\right\}$ for the high-types and $\left\{\overline{a}, \beta\delta R_l \overline{a}, (\beta\delta R_l)^2 \overline{a}, \ldots\right\}$ for the low-types.

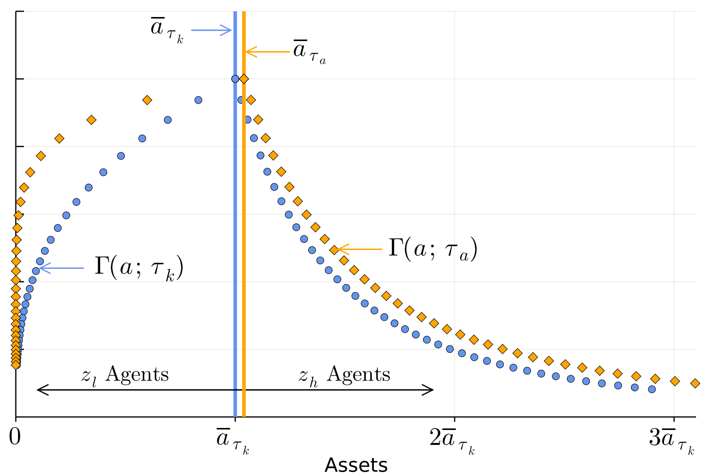
The share of entrepreneurs of type i with wealth $a = (\beta\delta R_i)^t \overline{a}$ is given by the share of agents who have lived exactly t periods:

$$\Gamma_i\left((\beta\delta R_i)^t \overline{a}\right) = \Pr(\text{age} = t) = \delta^t (1-\delta) \tag{55}$$

So the distribution of wealth is a geometric distribution with parameter δ.[16]

[16] The characterization of the stationary distribution of assets mimics the derivations in Jones (2015) adapted to the discrete time setting.

Figure 7: Stationary Distribution of Assets



Note: *The figure reports the stationary distribution of assets for two economies. The blue circles correspond to an economy with only capital income taxes ($\tau_k = \theta$ and $\tau_a = 0$) and its values are labeled with τ_k. The orange diamonds correspond to an economy with wealth taxes (τ_k set to satisfy Assumption 1) and its values are labeled with τ_a. The horizontal axis is presented in units of average assets in the capital income tax economy ($\overline{a}_{\tau_k}$).*

Figure 7 illustrates the behavior of the stationary distribution of assets. Agents are born with initial wealth $\overline{a}$ and save or dissave at constant rates depending on their productivity. A change in taxes affects the location of the mass-points of the distribution. In the figure, we contrast and economy without wealth taxes (that we label as τ_k) with one with wealth taxes (that we label as τ_a). The wealth tax economy has a higher level of overall wealth and hence $\overline{a}_{\tau_a}$ is to the right of $\overline{a}_{\tau_k}$. The change in $\overline{a}$ impacts all mass points (which are proportional to $\overline{a}$), shifting them rightwards. Then the increase in the dispersion of wealth is explained by the increase in the dispersion of returns, something reminiscent of the results in Lemma 3 and that we verify below for this economy.

Finally, we define a convenient measure of wealth concentration in the economy. Since wealth is determined by type and age, we can define the top wealth share as the fraction of wealth held by high types above an age t. This would correspond to the wealth share of the top $100 \times (1-\delta)\sum_{s=t}^{\infty}\delta^s = 100 \times \delta^t$ percent. Their total wealth is given by

$$A_{h,t} \equiv (1-\delta)\sum_{s=t}^{\infty}\left(\beta\delta^2 R_h\right)^s \overline{a} = \left(\beta\delta^2 R_h\right)^t A_h.$$

Then the top wealth shares are

$$s_{h,t} \equiv \frac{\left(\beta\delta^2 R_h\right)^t A_h}{K} = \left(\beta\delta^2 R_h\right)^t s_h. \tag{56}$$

After an increase in wealth taxes the dispersion of returns increases, this affects the distribution by shifting the mass points, although it does not affect the mass associated with each point, as shown in Figure 7. Because s_h and R_h increase with the wealth tax, the top wealth share $s_{h,t} = \left(\beta\delta^2 R_h\right)^t s_h$ increase with the wealth tax.

Lemma 6. ***(Top-Wealth Shares and Wealth Taxes)*** *For all $\tau_a < \overline{\tau}_a^p$, a marginal increase in wealth taxes increases the top-wealth-shares (56). The percentage increase in the wealth share is higher for higher wealth levels.*

Finally, having derived the stationary distribution of wealth allows us to study the welfare implications of wealth accumulation. We compute the average welfare gain by each type (denoting it as $\mathrm{CE}_{2,i}$) in the following way

$$\sum_a \left(V_k\left(a,i\right) + \frac{\log\left(1+\mathrm{CE}_{2,i}\right)}{1-\beta\delta}\right)\Gamma_k\left(a,i\right) \quad = \quad \sum_a V_a\left(a,i\right)\Gamma_a\left(a,i\right).$$

This average measure depends on the assets of each agent through their distribution, and thus captures the effects of higher capital accumulation triggered by the tax reform. Using the wealth distribution, we obtain:

$$\log\left(1+\mathrm{CE}_{2,i}\right) = \frac{1-\beta\delta^2}{\left(1-\delta\right)\left(1-\beta\delta\right)}\log\frac{R_{a,i}}{R_{k,i}} + \log\frac{K_a}{K_k}. \tag{57}$$

The utilitarian welfare gain for the whole population CE_2 is given by $\log\left(1+\mathrm{CE}_2\right) = \sum_i n_i \log\left(1+\mathrm{CE}_{2,i}\right)$. For entrepreneurs, $\mathrm{CE}_{2,i}$ welfare gains depend on the accumulation of capital in the economy. Interestingly, the effect of aggregate capital is the same for both types of entrepreneurs. This is because they both benefit from starting their lives at a higher level of initial assets (recall that $\overline{a} = K/2$) and their future asset levels are all proportional to their initial wealth. This makes it possible even for low-productivity entrepreneurs to benefit from the increase in wealth taxes if elasticity of output with respect to capital is sufficiently high. We summarize these results in the following Lemma:

Lemma 7. ***(Welfare Gain by Agent Type)*** *For all $\tau_a < \overline{\tau}_a^p$ in the perpetual youth model and under Assumption 1, for high-productivity entrepreneurs, $CE_{2,h} > CE_{1,h}$ always and $CE_{1,h} > 0$ after a marginal increase in wealth taxes (τ_a). For low-productivity entrepreneurs $CE_{1,l} < 0$ always and $CE_{2,l} > 0$ after a marginal increase in wealth taxes if and only if*

$$\xi_K \geq -\frac{1-\beta\delta^2}{\left(1-\delta\right)\left(1-\beta\delta\right)}\xi_{R_l}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ *is the elasticity of variable* x *with respect to* Z*.*

The effects for overall welfare are similar and change the optimal combination of taxes by including an additional source of welfare gains coming from higher capital accumulation, much like the results in Proposition 5. However, unlike in Section (3.2), taking wealth accumulation into account does not necessarily lead to higher optimal wealth taxes or lower $\underline{\alpha}$ and $\overline{\alpha}$ thresholds. Higher initial wealth increases the benefits from the reform (we again have $\xi_w = \xi_K = \alpha/1-\alpha$), but also increases the losses from the lower expected returns due to the compounding effect of returns on individual asset accumulation (which are suffered by the low-productivity entrepreneurs). In proposition (10) we characterize the optimal tax levels that maximizes CE_2.

Proposition 10. ***(Optimal*** $\boldsymbol{CE_2}$ ***Taxes)*** *In the perpetual youth model, if Assumption 1 holds there exist a unique tax combination* $\left(\tau^\star_{a,2}, \tau^\star_{k,2}\right)$ *that maximizes the utilitarian welfare measure* CE_2*, an interior solution* $\tau^\star_{a,2} < \overline{\tau}^p_a$ *is the solution to:*

$$n_w \xi_w + (1-n_w)\,\xi_K = \xi = -\,(1-n_w)\,\frac{1-\beta\delta^2}{(1-\delta)\,(1-\beta\delta)}\left(\frac{\xi_{R_l}+\xi_{R_h}}{2}\right) \tag{58}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ *is the elasticity of variable* x *with respect to* Z*. Furthermore, there exist two cutoff values for* α *which we denote* $\underline{\alpha}_2$ *and* $\overline{\alpha}_2$ *such that* $\left(\tau^\star_{a,2}, \tau^\star_{k,2}\right)$ *satisfies the following properties:*

$$\begin{aligned}
\tau^\star_{a,2} &\in \left[1-\frac{1}{\beta\delta}, 0\right) \text{ and } \tau^\star_{k,2} > \theta & &\text{if } \alpha < \underline{\alpha}_2 \\
\tau^\star_{a,2} &\in \left[0, \frac{\theta\,(1-\beta\delta)}{\beta\delta\,(1-\theta)}\right] \text{ and } \tau^\star_{k,2} \in [0,\theta] & &\text{if } \underline{\alpha}_2 \le \alpha \le \bar{\alpha}_2 \\
\tau^\star_{a,2} &\in \left(\frac{\theta\,(1-\beta\delta)}{\beta\delta\,(1-\theta)}, \tau^{\max}_{a,2}\right) \text{ and } \tau^\star_{k,2} < 0 & &\text{if } \alpha > \bar{\alpha}_2
\end{aligned}$$

where $\tau^{\max}_{a,2} \ge 1$*,* $\underline{\alpha}_2$ *and* $\overline{\alpha}_2$ *are the solutions to equation (58) with* $\tau_a = 0$ *and* $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$*, respectively. Recall from Lemma (3) that* $\xi = \alpha/1-\alpha$*.*

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APPENDIX

A Entrepreneur's problem

We start with an entrepreneur's labor choice given her capital:

$$\pi\left(z,k\right)=\max_{n}\left(zk\right)^{\alpha}n^{1-\alpha}-wn,$$

which gives the following labor demand

$$n^{\star}\left(z,k\right) = \left(\frac{1-\alpha}{w}\right)^{1/\alpha}zk. \tag{59}$$

Substituting the optimal labor demand into the profit, the entrepreneur's capital choice is given by

$$k^{\star}\left(z,a\right) = \arg\max_{0\leq k\leq\lambda a}\left[\alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha}z-r\right]k.$$

The optimal capital decision of the entrepreneur is characterized by the following function:

$$k^{\star}\left(z,a\right) = \begin{cases}\lambda a & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha}z>r\\ [0,\lambda a] & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha}z=r\\ 0 & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha}z<r.\end{cases}$$

Thus, entrepreneurs whose marginal return to capital is greater than the interest rate borrow up to the limit and sets λa and those whose return is below the interest rate does not produce zero output and earns the return r in the bond market on wealth a.

The optimal profit of the entrepreneur can be written as

$$\pi^{\star}\left(z\right)a = \begin{cases}\left(\alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha}z-r\right)\lambda a & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha}z>r\\ 0 & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha}z\leq r\end{cases} \tag{60}$$

Given taxes τ_a and τ_k and constant prices, an entrepreneur's optimal savings problem can be written as

$$V\left(a,z\right) = \max_{a'}\log\left(c\right)+\beta\sum_{z'}\Pi\left(z'\mid z\right)V\left(a',z'\right)$$

subject to

$$c+a' = R\left(z\right)a,$$

where $R\left(z\right)=1-\tau_a+\left(1-\tau_k\right)\left(r+\pi^{\star}\left(z\right)\right)$ as in the main text.

We solve the dynamic programming problem of the entrepreneur via guess and verify. To this end, we guess that the value function of an entrepreneur of type $i \in \{l, h\}$ has the form

$$V_i(a) = m_i + n \log(a),$$

where $m_l, m_h, n \in \mathbb{R}$ are coefficients. Under this guess the optimal savings choice of the entrepreneur is characterized by

$$\frac{1}{R_i a - a_i^{'}} = \frac{\beta n}{a_i^{'}}.$$

Solving for savings gives:

$$a_i^{'} = \frac{\beta n}{1+\beta n} R_i a.$$

Replacing the savings rule into the value function gives:

$$V_i(a) = \log\left(R_i a - a_i^{'}\right) + \beta\left(pV_i\left(a_i^{'}\right) + (1-p)V_j\left(a_i^{'}\right)\right)$$
$$m_i + n\log(a) = \log\left(R_i a - a_i^{'}\right) + \beta\left(pm_i + (1-p)m_j\right) + \beta n \log\left(a_i^{'}\right)$$
$$m_i + n\log(a) = \beta n \log(\beta n) + (1+\beta n)\log\left(\frac{R_i}{1+\beta n}\right) + \beta\left(pm_i + (1-p)m_j\right) + (1+\beta n)\log(a)$$

Matching coefficients:

$$n = 1 + \beta n$$
$$m_i = \beta n \log(\beta n) + (1+\beta n)\log\left(\frac{R_i}{1+\beta n}\right) + \beta\left(pm_i + (1-p)m_j\right),$$

where $j \neq i$. The solution to the first equation implies:

$$n = \frac{1}{1-\beta},$$

which in turn delivers the optimal saving decision of the entrepreneur:

$$a' = \beta R(z)\, a. \tag{61}$$

Finally, we solve for the remaining coefficients from the system of linear equations:

$$m_i = \frac{\beta}{1-\beta}\log\left(\frac{\beta}{1-\beta}\right) + \frac{1}{1-\beta}\log\left((1-\beta)R_i\right) + \beta\left(pm_i + (1-p)m_j\right)$$

The solution is given by:

$$m_i = \frac{\log(1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2}\log(\beta) + \frac{(1-\beta p)\log R_i + \beta(1-p)\log R_j}{(1-\beta)^2(1-\beta(2p-1))}$$

B Proofs

This appendix presents the proofs for the results listed in the paper. We reproduce the statement of all results for the reader's convenience.

Lemma 1. ***(Aggregate Variables in Equilibrium)*** *In the heterogenous return equilibrium* $((\lambda-1)A_h < A_l)$*, output, wages, interest rate, and gross returns on savings are:*

$$Y = (ZK)^{\alpha} L^{1-\alpha} \tag{62}$$
$$w = (1-\alpha)\left({}^{ZK}/_{L}\right)^{\alpha} \tag{63}$$
$$r = \alpha\left({}^{ZK}/_{L}\right)^{\alpha-1} z_l \tag{64}$$
$$R_l = (1-\tau_a) + (1-\tau_k)\,\alpha\left({}^{ZK}/_{L}\right)^{\alpha-1} z_l \tag{65}$$
$$R_h = (1-\tau_a) + (1-\tau_k)\,\alpha\left({}^{ZK}/_{L}\right)^{\alpha-1} z_{\lambda}. \tag{66}$$

Proof. We start by considering the labor market clearing condition

$$n^{\star}(z_h, K_h) + n^{\star}(z_l, K_l) = L.$$

Replacing for the optimal labor demand (59) we get

$$\begin{aligned}\left(\frac{1-\alpha}{w}\right)^{1/\alpha}(z_h K_h + z_l K_l) &= L\\ \left(\frac{1-\alpha}{w}\right)^{1/\alpha} Q &= L\end{aligned}$$

Manipulating this expression we get wages as:

$$w = (1-\alpha)\left({}^{Q}/_{L}\right)^{\alpha}. \tag{67}$$

Replacing into the equilibrium interest rate we get:

$$r = \alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z_l = \alpha\left({}^{Q}/_{L}\right)^{\alpha-1} z_l \tag{68}$$

These two expressions also let us rewrite the profit rate of the high-productivity entrepreneurs (from (5)):

$$\pi^{\star}(z_h) = \left(\alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z_h - r\right)\lambda = \alpha\left({}^{Q}/_{L}\right)^{\alpha-1}(z_h - z_l)\,\lambda \tag{69}$$

We can then use the equilibrium profit rates of entrepreneurs to rewrite the gross returns of entrepreneurs:

$$R_l = (1-\tau_a) + (1-\tau_k)\left(r + \pi^{\star}(z_l)\right)$$

and

$$\begin{aligned}R_h &= (1-\tau_a) + (1-\tau_k)\left(r + \pi^\star(z_h)\right)\\ &= (1-\tau_a) + (1-\tau_k)\,\alpha\left(Q/L\right)^{\alpha-1}\left(z_l + \lambda\left(z_h - z_l\right)\right)\\ &= (1-\tau_a) + (1-\tau_k)\,\alpha\left(Q/L\right)^{\alpha-1} z_\lambda\end{aligned}$$

Finally we consider aggregate output, for this note that the ratio of labor to capital is constant across entrepreneurs which allows us to aggregate in terms of the total capital of each type. From (59) we can express the output of an individual entrepreneur with productivity z and capital k as:

$$y(z,k) = \left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} zk = \left(Q/L\right)^{\alpha-1} zk,$$

where the second equality comes after replacing the wage from (67). Aggregate output is the sum of the total output produced by each type of entrepreneur:

$$\begin{aligned}Y &= \left(Q/L\right)^{\alpha-1}\left(z_h K_h + z_l K_l\right)\\ Y &= Q^\alpha L^{1-\alpha} \qquad (70)\end{aligned}$$

Alternatively we can write:

$$Y = (ZK)^\alpha L^{1-\alpha} \qquad (71)$$

This completes the derivation of the results.

□

Proposition 1. ***(Existence and Uniqueness of Steady State)*** *There exists a unique steady state. Moreover, the steady state equilibrium features heterogenous returns ($R_h > R_l$) if and only if*

$$\lambda < \overline{\lambda} \equiv 1 + \frac{(1-p)}{p - (2p-1)\left(\beta(1-\tau_a) + (1-\beta(1-\tau_a))\frac{z_l}{z_h}\right)}.$$

Proof. First, we show that the steady state is unique when $(\lambda - 1) A_h < A_l$. In this case, the steady state Z is the solution to equation (23). We will show that the larger root of that equation is the steady state Z. For this, let $h(z)$ be a function defined as

$$\begin{aligned}h(z) = (1-\beta(1-\tau_a)(2p-1))\,z^2 - (z_l + z_\lambda)(p - \beta(1-\tau_a)(2p-1))\,z&\\ + (2p-1)\,z_l z_\lambda(1-\beta(1-\tau_a)) &= 0.\end{aligned}$$

It is easy to show that $h(z_l) = (1-p)\,z_l(z_l - z_\lambda) < 0$ and $h(z_\lambda) = (1-p)\,z_\lambda(z_\lambda - z_l) > 0$. Since $h(z)$ is a quadratic function and $z_l < Z < z_\lambda$, this implies that there is a unique steady state Z as shown in Figure 1.

Next, we prove that $(\lambda - 1) A_h < A_l$ (excess supply of funds) iff $\lambda < \overline{\lambda}$ where

$$\overline{\lambda} \equiv 1 + \frac{(1-p)}{p - (2p-1)\left(\beta(1-\tau_a) + (1-\beta(1-\tau_a))\frac{z_l}{z_h}\right)}.$$

The proof involves two steps. First, we show that $(\lambda - 1) A_h < A_l$ iff $Z < z_h$. Second, we find the condition on λ so that $Z < z_h$. For the first step, substituting the definition of $Z = \frac{(z_h + (\lambda-1)(z_h - z_l))A_h + z_l A_l}{A_h + A_l}$ into $Z < z_h$ and some algebra gives $(\lambda - 1) A_h < A_l$. For the second step, we derive the condition on λ so that $h(z_h) > 0$ in equation (23). Thus to complete the proof, we evaluate $h(z_h)$:

$$\begin{aligned} h(z_h)/z_h^2 &= 1 - (2p-1)\beta(1-\tau_a) - \frac{(z_l + z_\lambda)}{z_h}(p - (2p-1)\beta(1-\tau_a)) \\ &+ (2p-1)\frac{z_l z_\lambda}{z_h^2}(1 - \beta(1-\tau_a)). \end{aligned}$$

Inserting $z_\lambda = z_h + (\lambda - 1)(z_h - z_l)$ gives

$$\begin{aligned} h(z_h)/z_h^2 &= 1 - (2p-1)\beta(1-\tau_a) - \frac{(z_l + z_h + (\lambda-1)(z_h - z_l))}{z_h}(p - (2p-1)\beta(1-\tau_a)) \\ &+ (2p-1)\frac{z_l(z_h + (\lambda-1)(z_h - z_l))}{z_h^2}(1 - \beta(1-\tau_a)). \end{aligned}$$

Next we combine the terms that include $\lambda - 1$:

$$\begin{aligned} h(z_h)/z_h^2 &= \frac{(1-p)(z_h - z_l)}{z_h} \\ &- \frac{(\lambda-1)(z_h - z_l)}{z_h}\left(p - (2p-1)\left(\beta(1-\tau_a) + (1-\beta(1-\tau_a))\frac{z_l}{z_h}\right)\right). \end{aligned}$$

Since $p - (2p-1)\left(\beta(1-\tau_a) + (1-\beta(1-\tau_a))\frac{z_l}{z_h}\right) > 0$ for all p, then, $h(z_h) > 0$ iff $\lambda - 1 < \frac{1-p}{p-(2p-1)\left(\beta(1-\tau_a)+(1-\beta(1-\tau_a))\frac{z_l}{z_h}\right)}$. Finally, recall that this equilibrium can only exist if $\lambda \leq 2$ (this gives $K_l \geq 0$). Inspecting the previous result it is immediate that $\overline{\lambda} \leq 2$ iff $p \geq 1/2$.

□

Corollary 1. ***(Savings Rates and Wealth Shares)*** *In steady state, the savings rate of type h entrepreneurs is positive and the savings rate of type l entrepreneurs is negative: $\beta R_h > 1 > \beta R_l$. Furthermore, $s_h > 1/2$ if and only if $p > 1/2$.*

Proof. The gross saving rate of the entrepreneurs is βR_i. We first show that $\beta R_i > 1$ if and only if $\overline{z}_i > Z$, where we slightly abuse notation by letting $\overline{z}_l = z_l$. The result follows immediately from expressing the savings rate in terms of Z by substituting R_i's from equation (22):

$$\begin{aligned} \beta R_i &> 1 \\ \beta(1-\tau_a) + (1-\beta(1-\tau_a))\,\overline{z}_i/Z &> 1 \\ \overline{z}_i &> Z \end{aligned}$$

To finalize the proof recall from Proposition 1 that the steady state Z satisfies $z_l < Z < z_\lambda$, this gives the desired result.

Now, consider $s_h \geq 1/2$. We know that $s_h = \frac{Z-z_l}{z_\lambda - z_l}$, so $s_h > 1/2$ is equivalent to $Z > \frac{z_\lambda + z_l}{2}$. We can verify if this is the case by evaluating the residual of (23) at $\frac{z_\lambda+z_l}{2}$:

$$\begin{aligned} h\left(\frac{z_\lambda+z_l}{2}\right) &= -(2p-1)(1-\beta(1-\tau_a))\left(\frac{z_\lambda+z_l}{2}\right)^2 + (2p-1)(1-\beta(1-\tau_a))\,z_l z_\lambda \\ &= -(2p-1)(1-\beta(1-\tau_a))\left[\left(\frac{z_\lambda+z_l}{2}\right)^2 - z_l z_\lambda\right] \\ &= -(2p-1)(1-\beta(1-\tau_a))\left(\frac{z_\lambda-z_l}{2}\right)^2 < 0 \end{aligned}$$

The residual is negative if and only if $p \geq 1/2$. So it must be that $Z > \frac{z_\lambda+z_l}{2}$ and thus $s_h > 1/2$ for $p \geq 1/2$.

□

Lemma 2. ***(Wealth Shares and Returns in Steady State)*** *The following equations and inequalities hold in steady state:*

$$s_h = \frac{1-\beta R_l}{\beta(R_h - R_l)} = \frac{Z - z_l}{z_\lambda - z_l} \qquad \frac{ds_h}{dZ} = \frac{1}{z_\lambda - z_l} > 0 \tag{72}$$

$$R_h = \frac{1}{\beta(2p-1)}\left(1 - \frac{1-p}{s_h}\right) \qquad \frac{dR_h}{dZ} > 0 \tag{73}$$

$$R_l = \frac{1}{\beta(2p-1)}\left(1 - \frac{1-p}{1-s_h}\right) \qquad \frac{dR_h}{dZ} < 0. \tag{74}$$

Moreover, the average are always decreasing with productivity, $\frac{d(R_l+R_h)}{dZ} < 0$*, while the geometric average of returns decreases if and only if* $p > 1/2$*,* $\frac{d(R_h R_l)}{dZ} < 0$*.*

Proof. Using equation (18) and imposing steady state, we obtain

$$A_l(1-\beta R_l) = (\beta R_h - 1) A_h, \tag{75}$$

Manipulating equation (75) directly we can express the ratio of wealth of the high types to total wealth:

$$\begin{aligned} A_l\left(1-\beta R_l\right) &= \left(\beta R_h-1\right)A_h \\ \left(1-\beta R_l\right)\left(A_l+A_h\right) &= \beta\left(R_h-R_l\right)A_h \\ \frac{A_h}{A_l+A_h} &= \frac{1-\beta R_l}{\beta\left(R_h-R_l\right)} \end{aligned} \tag{76}$$

The ratio depends on the returns of both types. We can further express the ratio in terms of Z by substituting R_i's from equation (22):

$$s_h=\frac{Z-z_l}{z_\lambda-z_l} \tag{77}$$

To finalize the proof take the derivative of s_h with respect to Z: $\frac{ds_h}{dZ}=\frac{1}{z_\lambda-z_l}>0$.

Now we consider what happens to R_h as Z increases. We start by considering the evolution equation for A_h in steady state (18)

$$\left(1-p\beta R_h\right)A_h \quad = \quad \left(1-p\right)\beta R_l A_l.$$

Manipulating this expression gives

$$R_h=\frac{1}{p\beta}-\left(\frac{1-p}{p}\right)\left(\frac{1-s_h}{s_h}\right)R_l.$$

We can also use the law of motion for A_l in steady state to obtain an expression for R_l in terms of R_h and s_h:

$$R_l=\frac{1}{p\beta}-\left(\frac{1-p}{p}\right)\left(\frac{s_h}{1-s_h}\right)R_h$$

Replacing we can solve for R_h as a function of s_h:

$$R_h=\frac{1}{\beta\left(2p-1\right)}\left(1-\frac{1-p}{s_h}\right) \tag{78}$$

We can now obtain the derivative of the high-type returns with respect to Z:

$$\frac{dR_h}{dZ}=\frac{1-p}{\beta\left(2p-1\right)}\frac{1}{s_h^2}\frac{ds_h}{dZ}>0 \tag{79}$$

The sign follows from Proposition 2.

We can also obtain an expression for R_l in terms of s_h:

$$R_l=\frac{1}{\beta\left(2p-1\right)}\left(1-\frac{1-p}{1-s_h}\right) \tag{80}$$

This expression allows to obtain an alternative expression for the derivative of the low-type returns with respect to Z:

$$\frac{dR_l}{dZ} = -\frac{(1-p)}{\beta(2p-1)}\frac{1}{(1-s_h)^2}\frac{ds_h}{dZ} < 0 \tag{81}$$

Using the results in (78), (79), (80), and (81) we can obtain expressions for the derivative of the sum and product of returns with respect to wealth taxes:

$$\frac{d(R_h+R_l)}{dZ} = \frac{-(2s_h-1)(1-p)}{\beta(2p-1)\left((1-s_h)^2 s_h^2\right)}\frac{ds_h}{dZ} \tag{82}$$

$$\frac{d(R_hR_l)}{dZ} = \frac{-(2s_h-1)p(1-p)}{[(1-s_h)s_h\beta(2p-1)]^2}\frac{ds_h}{dZ} \tag{83}$$

$\frac{d(R_h+R_l)}{dZ}$ is always negative because $s_h \geq 1/2$ if and only if $p \geq 1/2$ (see Corollary 2). $\frac{d(R_hR_l)}{dZ}$ is negative if and only if $s_h \geq 1/2$, again, this happens if and only if $p \geq 1/2$.

□

Lemma 3. ***(Aggregate Variables in Steady State)*** *Under Assumption 1, the steady state level of aggregate capital is*

$$K = \left(\frac{\alpha\beta(1-\theta)}{1-\beta}\right)^{\frac{1}{1-\alpha}} LZ^{\frac{\alpha}{1-\alpha}} \tag{84}$$

and the steady state elasticities of aggregate variables with respect to productivity are

$$\xi_K = \xi_Y = \xi_w = \xi \equiv \frac{\alpha}{1-\alpha} \qquad \text{and} \qquad \xi_Q = 1+\xi \tag{85}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Effective capital is $Q = ZK$, aggregate output is $Y = Q^{\alpha}L^{1-\alpha}$, and wage is $w = (1-\alpha)Y$ from Lemma 1. Moreover, the wealth levels of each entrepreneurial type in steady state are

$$A_h = \frac{Z-z_l}{z_\lambda - z_l}K \qquad \frac{dA_h}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}}(Z-\alpha z_l) > 0 \tag{86}$$

$$A_l = \frac{z_\lambda - Z}{z_\lambda - z_l}K \qquad \frac{dA_l}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}}(\alpha z_\lambda - Z) \tag{87}$$

where $\frac{dA_l}{dZ} < 0$ if and only if $\alpha z_\lambda < Z$.

Proof. Combining (21) with (24) gives $K = \left(\frac{\alpha\beta(1-\theta)}{1-\beta}\right)^{\frac{1}{1-\alpha}} LZ^{\frac{\alpha}{1-\alpha}}$. Inserting this into $Q = KZ$ gives Q, output Y and wage w follow from Lemma 1 The elasticity of of aggregate capital to productivity Z is

$$\xi_K \equiv \frac{d\log K}{d\log Z} = \frac{\alpha}{1-\alpha}$$

For convenience we define $\xi \equiv \alpha/1-\alpha$. The elasticities of output, wage, and effective capital with respect to productivity follow immediately.

From equation (77) we can express A_h in terms of Z and total capital K:

$$A_h = \frac{Z - z_l}{z_\lambda - z_l} K,$$

then we can replace K for its value in terms of Z to get

$$A_h = \frac{Z - z_l}{z_\lambda - z_l} \left(\frac{\alpha\beta\left(1-\theta\right)}{1-\beta} \right)^{\frac{1}{1-\alpha}} L Z^{\frac{\alpha}{1-\alpha}}. \tag{88}$$

The result follows from differentiating with respect to Z:

$$\frac{dA_h}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}} \left(Z - \alpha z_l\right) > 0,$$

where the inequality follows from $z_l < Z$.

A similar process lets us express A_l in terms of Z and total capital K:

$$A_l = \frac{z_\lambda - Z}{z_\lambda - z_l} K,$$

which gives

$$A_l = \frac{z_\lambda - Z}{z_\lambda - z_l} \left(\frac{\alpha\beta\left(1-\theta\right)}{1-\beta} \right)^{\frac{1}{1-\alpha}} L Z^{\frac{\alpha}{1-\alpha}}. \tag{89}$$

The result follows from differentiating with respect to Z:

$$\frac{dA_l}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}} \left[\alpha z_\lambda - Z\right]$$

which is negative if $\alpha z_\lambda < Z$.

□

Proposition 2. ***(Efficiency Gains from Wealth Taxation)*** *If the steady state equilibrium features return heterogeneity* $\left(\lambda < \overline{\lambda}\right)$, *aggregate productivity increases with wealth taxes,* $\frac{dZ}{d\tau_a} > 0$, *if and only if individual productivity shocks are positively correlated,* $p > 1/2$.

Proof. The steady state Z is given by the solution of $h\left(Z\right) = 0$ where $h\left(z\right)$ is defined in equation (23). Differentiating $h\left(z\right)$ with respect to τ_a gives

$$\begin{aligned} \frac{d}{d\tau_a} h\left(z\right) &= \left(2p-1\right)\beta z^2 - \left(2p-1\right)\beta\left(z_l + z_\lambda\right) z + \left(2p-1\right)\beta z_l z_\lambda \\ &= \left(2p-1\right)\beta z_l z_\lambda \left(z - z_l\right)\left(z - z_\lambda\right). \end{aligned}$$

We know that the steady state Z satisfies $z_l < Z < z_\lambda$, so we have $\left(z - z_l\right)\left(z - z_\lambda\right) < 0$. Thus, $\frac{d}{d\tau_a} h\left(z\right) < 0$ iff $p > 1/2$. As shown in Figure 3, the steady state Z increases when τ_a increases. Notice also that $\frac{d}{d\tau_a} h\left(z\right) < 0$ for all τ_a if $z_l < Z < z_\lambda$. Thus, $\frac{dZ}{d\tau_a} > 0$ for all τ_a as long as the economy is in the first equilibrium which happens if and only if $\lambda \leq \overline{\lambda}$. Notice that the bound $\overline{\lambda}$ is an increasing function of τ_a.

$\square$

Proposition 3. ***(Welfare Gain by Agent Type)*** *Under Assumption 1 and if $p > 1/2$, any increase in wealth taxes increases the welfare of workers ($CE_{1,w} > 0$) and decreases the welfare of low-productivity entrepreneurs ($CE_{1,l} < 0$) and the average welfare of entrepreneurs ($CE_1^e < 0$). Furthermore, there exists a upper bound on the dispersion of returns (κ_R) such that an increase in wealth taxes increases the welfare of high-productivity entrepreneurs ($CE_{1,h} > 0$) if and only if $R_h - R_l < \kappa_R$.*

Proof. We start by stating the welfare gain measure for each type of agent as in (34):

$$\log\left(1+\text{CE}_{1,i}\right) = \begin{cases} \log {w_a}/{w_k} & \text{if } i = w \\ \frac{(1-\beta)\log {R_{a,i}}/{R_{k,i}} + \beta(1-p)\left(\log {R_{a,l}}/{R_{k,l}} + \log {R_{a,h}}/{R_{k,h}}\right)}{(1-\beta)(1-\beta(2p-1))} & \text{if } i \in \{l,h\}\,. \end{cases}$$

For the workers' welfare note that:

$$\frac{d\log\left(1+\text{CE}_{1,w}\right)}{d\tau_a} = \frac{d\frac{\alpha}{1-\alpha}\log\left(Z_a/Z_k\right)}{d\tau_a} = \frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a} > 0 \longleftrightarrow p > 1/2$$

The welfare gain is positive if and only if productivity is persistent because of Proposition (2).

The welfare of the low-productivity entrepreneurs decreases unambiguously:

$$\frac{d\log\left(1+\text{CE}_{1,l}\right)}{d\tau_a} \propto \frac{1-\beta}{R_l}\frac{dR_l}{d\tau_a} + \frac{\beta\left(1-p\right)}{R_l R_h}\frac{dR_l R_h}{d\tau_a} < 0$$

which follows from Corollary (3) $\left(\frac{dR_l}{d\tau_a}, \frac{dR_l R_h}{d\tau_a} < 0\right)$.

The welfare of entrepreneurs as a group also decreases unambiguously.

$$\frac{d\log\left(1+\text{CE}_1^{\text{e}}\right)}{d\tau_a} = \frac{\beta\left(1-p\right)}{1-\beta}\frac{1}{R_l R_h}\frac{dR_l R_h}{d\tau_a} < 0$$

Finally, for the high-productivity entrepreneurs:

$$\begin{aligned} \frac{d\log\left(1+\text{CE}_{1,h}\right)}{d\tau_a} &\propto \frac{1-\beta}{R_h}\frac{dR_h}{d\tau_a} + \frac{\beta\left(1-p\right)}{R_l R_h}\frac{dR_l R_h}{d\tau_a} \\ &= \left[\left(1-\beta\right) - \frac{1}{R_l}\frac{\left(2s_h-1\right)p\left(1-p\right)}{\left(1-s_h\right)^2\left(2p-1\right)}\right]\frac{\left(1-p\right)}{\beta\left(2p-1\right)s_h^2 R_h}\frac{ds_h}{d\tau_a} \\ &= \left[\left(1-\beta\right) - \frac{\beta\left(2s_h-1\right)p\left(1-p\right)}{\left(p-s_h\right)\left(1-s_h\right)}\right]\frac{\left(1-p\right)}{\beta\left(2p-1\right)s_h^2 R_h}\frac{ds_h}{d\tau_a} \end{aligned}$$

We maintain the assumption that $p \geq 1/2$, and from Corollary 3 we know that $\frac{ds_h}{d\tau_a} > 0$. So, the sign of derivative of interest depends on the sign of the term in square brackets.

$$\frac{d \log \left(1 + \mathrm{CE}_{1,h}\right)}{d\tau_a} \geq 0 \longleftrightarrow 1 - \beta \geq \frac{\beta \left(2s_h - 1\right) p \left(1 - p\right)}{\left(p - s_h\right)\left(1 - s_h\right)}$$

It is easy to verify that in steady state $s_h < p$, which together with Corollary 2 implies that the right hand side of the inequality is always positive. To verify that $s_h < p$ holds in steady steady note that this condition is equivalent to $Z < pz_\lambda + (1-p) z_l$, then evaluate function h defined in (23) at $pz_\lambda + (1-p) z_l$. The value of h (the residual of the quadratic equation) is always positive, so it must be that $Z < pz_\lambda + (1-p) z_l$ and thus $s_h < p$.

Then, the high-type entrepreneurs' welfare gain is positive if and only if

$$g\left(s_h\right) \equiv \left(1-\beta\right)\left(p - s_h\right)\left(1 - s_h\right) - \beta\left(2s_h - 1\right) p \left(1-p\right) \geq 0. \tag{90}$$

Evaluating at $s_h = 1/2$

$$g\left(s_h\right) = \left(1-\beta\right)\left(p - \frac{1}{2}\right)\frac{1}{2} > 0.$$

Evaluating at $s_h = p$

$$g\left(s_h\right) \equiv -\beta\left(2p-1\right) p \left(1-p\right) < 0.$$

Moreover, g is continuous for $s_h \in [1/2, p]$ and monotonically decreasing:

$$g^{'}\left(s_h\right) = -\left(1-\beta\right)\left[\left(1-s_h\right) + \left(p - s_h\right)\right] - 2\beta p \left(1-p\right) < 0$$

So, there exists an upper bound $\bar{s}_h$ such that

$$\frac{d \log \left(1 + \mathrm{CE}_{1,h}\right)}{d\tau_a} \geq 0 \longleftrightarrow s_h \in \left[\frac{1}{2}, \bar{s}_h\right]$$

The upper bound for z_l is characterized by the solution to

$$\left(p - \bar{s}_h\right)\left(1 - \bar{s}_h\right) - \beta\left(2\bar{s}_h - 1\right) p \left(1-p\right) = 0$$

Alternatively, we can make us of the link between s_h and the dispersion of returns:

$$R_h - R_l = \frac{\left(1-p\right)\left(2s_h - 1\right)}{\beta\left(2p-1\right)\left(1-s_h\right) s_h}$$

So the high-productivity entrepreneurs benefit from an increase in wealth taxes if and only if the dispersion of returns is low enough:

$$\frac{d \log \left(1 + \mathrm{CE}_{1,h}\right)}{d\tau_a} \geq 0 \longleftrightarrow s_h \in \left[\frac{1}{2}, \bar{s}_h\right] \longleftrightarrow R_h - R_l \in \left[0, \kappa_R\right]$$

where $\kappa_R \equiv \frac{(1-p)(2\bar{s}_h - 1)}{\beta(2p-1)(1-\bar{s}_h)\bar{s}_h}$. Note that $\bar{s}_h$ depends only on p and β, therefore the upper bound for the dispersion of returns is also a function of p and β alone.

□

Proposition 4. ***(Optimal*** CE_1 ***Taxes)*** *Under Assumption 1 and if* $p > 1/2$*, there exist a unique tax combination* (τ_a^*, τ_k^*) *that maximizes the utilitarian welfare measure* CE_1 *and given by the solution to the following equation:*

$$n_w \xi_w = -\frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_l}+\xi_{R_h}}{2}\right) \tag{91}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ *is the elasticity of variable* x *with respect to* Z*. Furthermore, there exist two cutoff values for* α *which we denote* $\underline{\alpha}$ *and* $\overline{\alpha}$ *such that* $(\tau_a^\star, \tau_k^\star)$ *satisfies the following properties:*

$$\begin{aligned}
\tau_a^\star &\in \left[1-\frac{1}{\beta}, 0\right) \text{ and } \tau_k^\star > \theta && \text{if } \alpha < \underline{\alpha} \\
\tau_a^\star &\in \left[0, \frac{\theta(1-\beta)}{\beta(1-\theta)}\right] \text{ and } \tau_k^\star \in [0, \theta] && \text{if } \underline{\alpha} \le \alpha \le \bar{\alpha} \\
\tau_a^\star &\in \left(\frac{\theta(1-\beta)}{\beta(1-\theta)}, \tau_a^{\max}\right) \text{ and } \tau_k^\star < 0 && \text{if } \alpha > \bar{\alpha}
\end{aligned}$$

where $\underline{\alpha}$ *and* $\overline{\alpha}$ *are the solutions to equation (91) with* $\tau_a = 0$ *and* $\tau_a = \tau^{TR} = \frac{\theta(1-\beta)}{\beta(1-\theta)}$*, respectively. Recall from Lemma (3) that* $\xi_w = \xi \equiv \alpha/1-\alpha$*.*

Proof. We start from the definition of aggregate CE_1 welfare which gives us:

$$\text{CE}_1 > 0 \longleftrightarrow \sum_{i\in\{w,h,l\}} n_i \log(1+\text{CE}_{1,i}) > 0$$

replacing from (34) gives us:

$$\sum_{i\in\{w,h,l\}} n_i \log(1+\text{CE}_{1,i}) = n_w \log\frac{w_a}{w_k} + \frac{1-n_w}{2(1-\beta)}\log\frac{R_{a,l}R_{a,h}}{R_{k,l}R_{k,h}}$$

The optimal tax is characterized by first order condition:

$$\begin{aligned}
n_w \frac{d\log w}{d\tau_a} + \frac{1-n_w}{2(1-\beta)}\frac{d\log R_l R_h}{d\tau_a} &= 0 \\
\left[n_w \frac{d\log w}{d\log Z} + \frac{1-n_w}{2(1-\beta)}\frac{d\log R_l R_h}{d\log Z}\right]\frac{d\log Z}{d\tau_a} &= 0 \\
\left[n_w \xi_w + \frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_l}+\xi_{R_h}}{2}\right)\right]\frac{d\log Z}{d\tau_a} &= 0
\end{aligned}$$

From Proposition (2) we know that $\frac{d\log Z}{d\tau_a} > 0$ under the sustained assumptions that $p > 1/2$ and

$\lambda < \overline{\lambda}$. Then the above equation is satisfied if and only if

$$n_w \xi_w = -\frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_l}+\xi_{R_h}}{2}\right)$$

That is, for τ_a such that the steady state values of the elasticities above satisfy the equation. The elasticity of wages with respect to productivity is constant, while the average elasticity of returns is negative (because the geometric average of returns decreases with taxes). Further, we can show that the average elasticity of returns is increasing in wealth taxes (this follows immediately from the explicit solution below). So there exists at most one solution to the optimal wealth taxes.

Note that elasticity of wages depends only on α, while the elasticities of returns are independent of α. Because of this we can define cutoffs for α such by evaluating the right hand side of the equation at $\tau_a = 0$ and $\tau_a = \frac{\theta(1-\beta)}{\beta(1-\theta)}$. If α is exactly equal to the cutoff then the optimal τ_a is either 0 or $\frac{\theta(1-\beta)}{\beta(1-\theta)}$. The monotonicity of the right hand side lets us define the intervals shown in the proposition.

Finally, we can replace to get a more explicit solution using Lemmas 3 and 3:

$$n_w\frac{\alpha}{1-\alpha} = -\frac{1-n_w}{2(1-\beta)}\frac{Z}{R_lR_h}\frac{dR_lR_h}{dZ}$$
$$n_w\frac{\alpha}{1-\alpha} = \frac{1-n_w}{2(1-\beta)}\frac{Z}{R_lR_h}\frac{(2s_h-1)\,p\,(1-p)}{[(1-s_h)\,s_h\beta\,(2p-1)]^2}\frac{ds_h}{dZ}$$
$$n_w\frac{\alpha}{1-\alpha} = \frac{1-n_w}{2(1-\beta)}\frac{1}{R_lR_h}\frac{(2s_h-1)\,p\,(1-p)}{[(1-s_h)\,s_h\beta\,(2p-1)]^2}\frac{Z}{(z_\lambda-z_l)}$$
$$n_w\frac{\alpha}{1-\alpha} = \frac{1-n_w}{2(1-\beta)}\frac{(2s_h-1)\,p\,(1-p)}{(p-s_h)\,(p+s_h-1)\,(1-s_h)\,s_h}\frac{Z}{(z_\lambda-z_l)}$$

□

Corollary 4. ***(α-Thresholds)*** *If $z_l = 0$, the thresholds $\underline{\alpha}$ and $\bar{\alpha}$ are explicitly given by* $\frac{\underline{\alpha}}{1-\underline{\alpha}} = \frac{1}{L}\frac{p}{1-p}\frac{(1-\beta(2p-1))^2}{\beta(2p-1)(1-p\beta)}$ *and* $\frac{\bar{\alpha}}{1-\bar{\alpha}} = \frac{p(1-\theta-(2p-1)(\beta-\theta))^2}{L(1-p)(2p-1)(1-\theta)(\beta-\theta)((1-\theta)-p(\beta-\theta))}$.

Proof. When $z_l = 0$ we can solve for Z and s_h explicitly as:

$$Z = \frac{z_\lambda\,(p-(1-\tau_a)\,\beta\,(2p-1))}{1-(1-\tau_a)\,\beta\,(2p-1)} \qquad s_h = \frac{Z}{z_\lambda}$$

The value of $\underline{\alpha}$ is obtained when $\tau_a = 0$, so $Z = \frac{z_\lambda(p-\beta(2p-1))}{1-\beta(2p-1)}$ and $s_h = \frac{p-\beta(2p-1)}{1-\beta(2p-1)}$. We can then evaluate the expression:

$$\begin{aligned}\frac{\underline{\alpha}}{1-\underline{\alpha}} &= \frac{1-n_w}{n_w}\frac{Z}{2(1-\beta)}\frac{(2s_h-1)\,p\,(1-p)}{(p-s_h)(p+s_h-1)(1-s_h)\,s_h}\frac{1}{z_\lambda}\\ &= \frac{1}{L}\frac{1}{(1-\beta)}\frac{(2s_h-1)\,p\,(1-p)}{(p-s_h)(p+s_h-1)(1-s_h)}\\ &= \frac{1}{L}\frac{p}{1-p}\frac{(1-\beta(2p-1))^2}{\beta(2p-1)(1-p\beta)}\end{aligned}$$

The value of $\overline{\alpha}$ is obtained when $\tau_a = \frac{\theta(1-\beta)}{\beta(1-\theta)}$, so $Z = \frac{z_\lambda((1-\theta)p-(\beta-\theta)(2p-1))}{(1-\theta)-(\beta-\theta)(2p-1)}$ and $s_h = \frac{(1-\theta)p-(\beta-\theta)(2p-1)}{(1-\theta)-(\beta-\theta)(2p-1)}$. We can then evaluate the expression:

$$\begin{aligned}\frac{\overline{\alpha}}{1-\overline{\alpha}} &= \frac{1-n_w}{n_w}\frac{Z}{2(1-\beta)}\frac{(2s_h-1)\,p\,(1-p)}{(p-s_h)(p+s_h-1)(1-s_h)\,s_h}\frac{1}{z_\lambda}\\ &= \frac{1}{L}\frac{1}{(1-\beta)}\frac{(2s_h-1)\,p\,(1-p)}{(p-s_h)(p+s_h-1)(1-s_h)}\\ &= \frac{1}{L}\frac{p}{1-p}\frac{1}{(2p-1)(1-\theta)(\beta-\theta)}\frac{((1-\theta)-(\beta-\theta)(2p-1))^2}{((1-\theta)-p(\beta-\theta))}.\end{aligned}$$

We can also compute the optimal wealth tax for this case inserting $z_l = 0$, $Z = \frac{z_\lambda(p-(1-\tau_a)\beta(2p-1))}{1-(1-\tau_a)\beta(2p-1)}$, and $s_h = \frac{p-(1-\tau_a)\beta(2p-1)}{1-(1-\tau_a)\beta(2p-1)}$ into

$$n_w\frac{\alpha}{1-\alpha} = \frac{1-n_w}{2(1-\beta)}\frac{(2s_h-1)\,p\,(1-p)}{(p-s_h)(p+s_h-1)(1-s_h)\,s_h}\frac{Z}{(z_\lambda-z_l)},$$

which gives the following non-linear equation which uniquely determines the optimal wealth tax as functions of parameters:

$$\frac{n_w}{(1-n_w)/2}\frac{\alpha}{1-\alpha}(1-\tau_a) = \frac{p\,(1-(1-\tau_a)\beta)\,(1-(1-\tau_a)\beta(2p-1))^2}{\beta(1-\beta)(1-p)(2p-1)(1-p(1-\tau_a)\beta)}.$$

□

Lemma 4. ***(Welfare Gains by Agent Type with Asset Accumulation)*** *Under Assumption 1 and if $p > 1/2$, any increase in wealth taxes increases the welfare of high-productivity entrepreneurs ($CE_{2,h} > 0$). The welfare of low-productivity entrepreneurs decreases ($CE_{2,l} < 0$) if $\alpha z_\lambda < Z$.*

Proof. The welfare gain of the average low-productivity entrepreneur is:

$$\log(1+\mathrm{CE}_{2,l}) = (1-\beta)(V_a(A_{l,a},l) - V_k(A_{l,k},l)) = \log(1+\mathrm{CE}_{1,l}) + \log(A_{a,l}/A_{k,l})$$

From Lemma 3 we know that $\mathrm{CE}_{1,l} < 0$ and from Lemma 3 we know that $A_{a,l} < A_{k,l}$ if $\alpha z_\lambda < Z$.

Now, we turn to the CE_2 welfare measure for the high-type entrepreneurs:

$$\log\left(1+\text{CE}_{2,h}\right)=(1-\beta)\left(V_a\left(A_{h,a},h\right)-V_k\left(A_{h,k},h\right)\right)=\log\left(1+\text{CE}_{1,h}\right)+\log\left({}^{A_{a,h}}/_{A_{k,h}}\right)$$

Substituting $A_h = s_h K$ and substituting K from Lemma 3 and taking derivative with respect to τ_a gives:

$$\begin{aligned}
\frac{d\log\left(1+\text{CE}_{2,h}\right)}{d\tau_a} &\propto \frac{1-\beta}{R_h}\frac{dR_h}{d\tau_a}+\frac{\beta(1-p)}{R_lR_h}\frac{dR_lR_h}{d\tau_a}+\frac{1}{s_h}\frac{ds_h}{d\tau_a}+\frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a}\\
&=\left[\frac{1-\beta}{R_h}\left(\frac{1-p}{\beta(2p-1)}\frac{1}{s_h^2}\right)-\frac{\beta(1-p)}{R_lR_h}\left(\frac{(2s_h-1)\,p\,(1-p)}{\left[(1-s_h)\,s_h\beta\,(2p-1)\right]^2}\right)\right]\frac{ds_h}{d\tau_a}\\
&\qquad+\frac{1}{s_h}\frac{ds_h}{d\tau_a}+\frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a}\\
&=\left[\frac{(1-\beta)(1-p)}{(p+s_h-1)}-\frac{\beta(2s_h-1)\,p\,(1-p)^2}{(p-s_h)(s_h+p-1)(1-s_h)}+1\right]\frac{1}{s_h}\frac{ds_h}{d\tau_a}+\frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a}\\
&=\left[\left(s_h-\beta(1-p)\,\frac{2p-s_h+s_h^2-3ps_h+(2s_h-1)\,p^2}{(p-s_h)(1-s_h)}\right)\frac{1}{(s_h+p-1)(Z-z_l)}\right.\\
&\qquad\left.+\frac{\alpha}{1-\alpha}\frac{1}{Z}\right]
\end{aligned}$$

A sufficient condition for this to be positive is that

$$s_h>\beta(1-p)\left(\frac{2p-s_h+s_h^2-3ps_h+(2s_h-1)\,p^2}{(p-s_h)(1-s_h)}\right)$$

We know that $s_h < p$ and that $s_h \geq 1/2$. So a sufficient condition is

$$\begin{aligned}
\frac{1}{2}&>\beta(1-p)\left(\frac{2p-s_h+s_h^2-3ps_h+(2s_h-1)\,p^2}{(p-s_h)(1-s_h)}\right)\\
\frac{1}{2}&>\beta(1-s_h)\left(\frac{2p-s_h+s_h^2-3ps_h+(2s_h-1)\,p^2}{(p-s_h)(1-s_h)}\right)\\
p-s_h&>4p-2s_h+2s_h^2-6ps_h+2\,(2s_h-1)\,p^2\\
0&>3p\,(1-2s_h)-s_h\,(1-2s_h)+2\,(2s_h-1)\,p^2\\
0&>-\left[2p\,(1-p)+(p-s_h)\right](2s_h-1)
\end{aligned}$$

which is verified for all values of $p > s_h > 1/2$.

□

Proposition 5. ***(Optimal $\widetilde{CE}_2$ Taxes)*** *Under Assumption 1 and if $p > 1/2$, there exist a unique tax combination $\left(\tau_{a,2}^*,\tau_{k,2}^*\right)$ that maximizes the utilitarian welfare measure $\widetilde{CE}_2$ and given by the solution to the following equation:*

$$n_w\xi_w+(1-n_w)\,\xi_K=\xi=-\frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_l}+\xi_{R_h}}{2}\right) \tag{92}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ *is the elasticity of variable* x *with respect to* Z*. Furthermore, there exist two cutoff values for* α *which we denote* $\underline{\alpha}_2$ *and* $\overline{\alpha}_2$ *such that* $\left(\tau^\star_{a,2}, \tau^\star_{k,2}\right)$ *satisfies the following properties:*

$$
\begin{aligned}
\tau^\star_{a,2} &\in \left[1-\frac{1}{\beta}, 0\right) \text{ and } \tau^\star_{k,2} > \theta && \text{if } \alpha < \underline{\alpha}_2 \\
\tau^\star_{a,2} &\in \left[0, \frac{\theta(1-\beta)}{\beta(1-\theta)}\right] \text{ and } \tau^\star_{k,2} \in [0,\theta] && \text{if } \underline{\alpha}_2 \le \alpha \le \bar{\alpha}_2 \\
\tau^\star_{a,2} &\in \left(\frac{\theta(1-\beta)}{\beta(1-\theta)}, \tau^{\max}_{a,2}\right) \text{ and } \tau^\star_{k,2} < 0 && \text{if } \alpha > \bar{\alpha}_2
\end{aligned}
$$

where $\underline{\alpha}_2$ *and* $\overline{\alpha}_2$ *are the solutions to equation (92) with* $\tau_a = 0$ *and* $\tau_a = \frac{\theta(1-\beta)}{\beta(1-\theta)}$*, respectively. Recall from Lemma (3) that* $\xi = \alpha/1-\alpha$*.*

Proof. From (38) we obtain the first order condition to maximize $\widetilde{\text{CE}}_2$:

$$
\begin{aligned}
\frac{d\log(1+\text{CE}_1)}{d\tau_a} + (1-n_w)\frac{d\log K}{d\tau_a} &= 0 \\
\left[\frac{d\log(1+\text{CE}_1)}{d\log Z} + (1-n_w)\frac{d\log K}{d\log Z}\right]\frac{d\log Z}{d\tau_a} &= 0 \\
\left[n_w\xi_w + \frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_l}+\xi_{R_h}}{2}\right) + (1-n_w)\xi_K\right]\frac{d\log Z}{d\tau_a} &= 0
\end{aligned}
$$

As in the proof of Proposition 4 this leads to the optimality condition:

$$
n_w\xi_w + (1-n_w)\xi_K = -\frac{1-n_w}{1-\beta}\left(\frac{\xi_{R_l}+\xi_{R_h}}{2}\right).
$$

Further, we know from Lemma 3 that $\xi_w = \xi_K = \xi = \alpha/1-\alpha$. The right hand side of the equation is the same as in Proposition 4 and an explicit formula can be found in the proof to that proposition. The uniqueness of the solution and the definition of the thresholds for α and its implications for the optimal taxes follow from the same arguments as in Proposition 4.

□

C Extensions

C.1 Corporate sector

Consider a model like that in Section 2 where there is also a corporate sector that produces the same final good as the entrepreneurs using a constant returns to scale technology:

$$Y_c = (z_c K_c)^{\alpha} L_c^{1-\alpha}. \tag{93}$$

The conditional demand for labor of the corporate sector is characterized by

$$w = (1-\alpha)\left(\frac{z_c K_c}{L_c}\right)^{\alpha}$$

as is standard.

Unlike the entrepreneurs, the corporate sector faces no collateral constraints and thus the demand for capital is:

$$K_c = \begin{cases} \infty & \text{if } \alpha z_c \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} > r \\ [0,\infty) & \text{if } \alpha z_c \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} = r \\ 0 & \text{if } \alpha z_c \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} < r \end{cases}$$

So, any equilibrium must satisfy:

$$r \leq \alpha z_c \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}.$$

If $r < \alpha z_c \left({}^{1-\alpha}/_{w}\right)^{1-\alpha/\alpha}$ the corporate sector does not operate and the economy works as in Section 2. This happens if z_c is too low relative to the productivity of entrepreneurs, $z_c < z_l$. The more interesting case is when $r = \alpha z_c \left({}^{1-\alpha}/_{w}\right)^{1-\alpha/\alpha}$ in equilibrium which happens if $z_l \leq z_c$. The behavior of the entrepreneurs depends then on how high the corporate sector's productivity is relative to that of the entrepreneurs. We will focus on the most relevant scenario where $z_l < z_c < z_h$. In this scenario both the corporate sector and the high-productivity entrepreneurs operate in equilibrium, while the low-productivity entrepreneurs do not produce and instead lend all of their funds.[17]

Even though the corporate sector is operating in equilibrium, there are no real changes in the aggregates of the economy. In fact, the equilibrium looks just like that of Section 2 with the corporate sector's productivity z_c taking the place of the z_l. As in Section 2, the high-productivity entrepreneurs are constrained in their demand for capital and demand $K_h = \lambda A_h$, but now the

[17] If $z_c > z_h$, only the corporate sector operates in equilibrium and it is optimal for all entrepreneurs to lend their assets to the corporate sector where they will receive a higher return. The equilibrium is efficient and total productivity Z is equal to z_c. The knife-edge case with $z_c = z_h$ has the same result but the distribution of capital between the high-productive entrepreneurs and the corporate sector is indeterminate. Finally, the knife-edge case with $z_l = z_c$ is identical to the model in Section 2 with the low-productivity entrepreneurs being indifferent between producing themselves or lending to the corporate sector. All aggregates remain unchanged.

remaining capital is used by the corporate sector rather than by the low-productivity entrepreneurs. This is only sustainable in equilibrium if $r = \alpha z_c \left(1-\alpha/w\right)^{1-\alpha/\alpha}$ as noted above, and in this way z_c takes the place of z_l in determining the interest rate in the economy. The main consequence of this change is that the relevant dispersion of productivities is now $z_h - z_c$ which is lower than it was in Section 2 (recall that $z_c > z_l$). This reduces the range of parameters for which the heterogeneous return equilibrium applies, and reduces the scope for misallocation and thus for efficiency gains. Lemma 5 makes the above results precise:

Lemma 5. *If $z_l < z_c < z_h$ and*

$$\lambda < \lambda_c^\star \equiv 1 + \frac{(1-p)}{p - (2p-1)\left(\beta(1-\tau_a) + (1-\beta(1-\tau_a))\frac{z_c}{z_h}\right)} < \overline{\lambda},$$

the economy is in the heterogeneous return equilibrium with both the corporate sector and high-productivity entrepreneurs operating and output, wages, interest rate, and gross returns on savings are the same as in Lemma 1 with z_c taking the place of z_l:

$$Y = (ZK)^\alpha L^{1-\alpha} \tag{94}$$

$$w = (1-\alpha)\left(ZK/L\right)^\alpha \tag{95}$$

$$r = \alpha\left(ZK/L\right)^{\alpha-1} z_c \tag{96}$$

$$R_l = (1-\tau_a) + (1-\tau_k)\,\alpha\left(ZK/L\right)^{\alpha-1} z_c \tag{97}$$

$$R_h = (1-\tau_a) + (1-\tau_k)\,\alpha\left(ZK/L\right)^{\alpha-1} z_\lambda. \tag{98}$$

where $Z \equiv s_h z_\lambda + s_l z_c$, $z_\lambda \equiv z_h + (\lambda-1)(z_h - z_c)$ and $s_h = A_h/K$.

Proof. If $z_l < z_c < z_h$, $\alpha z_l \left(1-\alpha/w\right)^{1-\alpha/\alpha} < r = \alpha z_c \left(1-\alpha/w\right)^{1-\alpha/\alpha} < \alpha z_h \left(1-\alpha/w\right)^{1-\alpha/\alpha}$ and thus $K_l = 0$, $K_h = \lambda A_h$, and $K_c = A_l - (\lambda-1)A_h$ to guarantee that the capital market clears.

Given the wage rate w, the labor demand of the corporate and private sectors are:

$$n_i^\star(K_i) = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} z_i K_i\,; \quad i \in \{l,h,c\}\,.$$

The labor market clearing condition gives

$$\left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}}(z_h K_h + z_l K_l + z_c K_c) = L$$

$$\left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}} Q = L$$

$$(1-\alpha)\left(\frac{Q}{L}\right)^{\alpha} = w$$

where $Q = z_h K_h + z_l K_l + z_c K_c = ZK$ with $Z = s_h z_\lambda + s_l z_c$ after replacing for the equilibrium capital demand. Then, the equilibrium interest rate is

$$r = \alpha z_c \left(\frac{Q}{L}\right)^{\alpha-1}.$$

In equilibrium, low productivity entrepreneurs and the corporate sector do not generate profits, while high-productivity entrepreneurs do:

$$\pi^\star (z_h) = \left(\alpha \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z_h - r\right) \lambda = \alpha \left(Q/L\right)^{\alpha-1} (z_h - z_c) \lambda$$

Total private output corresponds to the output of high-productivity entrepreneurs. Note that the output of an individual entrepreneur is proportional to their capital, so total private output is:

$$Y_p = Y_h = \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z_h K_h$$

Total output is then:

$$\begin{aligned} Y \equiv Y_c + Y_p &= \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} (z_c K_c + z_h K_h) \\ &= Q^\alpha L^{1-\alpha} \end{aligned}$$

Finally we derive the after-tax returns on savings for low- and high-productivity entrepreneurs. Low-productivity entrepreneurs do not produce so they have:

$$R_l = (1-\tau_a) + (1-\tau_k) \underbrace{\alpha \left(Q/L\right)^{\alpha-1} z_c}_{r}.$$

High productivity entrepreneurs have:

$$R_h = (1-\tau_a) + (1-\tau_k) \underbrace{\alpha \left(Q/L\right)^{\alpha-1} z_\lambda}_{r+\pi^\star(z_h)}$$

where $z_\lambda \equiv z_h + (\lambda - 1)(z_h - z_c)$.

All aggregates are then as in Lemma 1 with z_c taking the role of z_c. Consequently, Proposition 1 applies with the only modification of z_c replacing z_l in the condition that characterizes the steady state value of Z and the upper bound for λ.

□

C.2 Entrepreneurial effort

Consider a model like that in Section 2 where entrepreneurs can exert effort to increase their productivity. We capture the effect of effort as modifying the production function of entrepreneurs to:

$$y = (zk)^{\alpha} g(e)^{\gamma} n^{1-\alpha-\gamma}. \tag{99}$$

where $\gamma \in [0, 1)$.

Exerting effort has a utility cost of $h(e)$, where $h^{'}(e) > 0$ and $h^{''}(e) \geq 0$ but no dynamic effects. The utility function is

$$u(c, e) = \log(c - h(e)).$$

C.2.1 Entrepreneur's problem

We can solve the entrepreneur's static effort choice. The solution is characterized by the following first order conditions:

$$u_e h^{'}(e) = (1-\tau_k) u_c \cdot \gamma (zk)^{\alpha} g(e)^{\gamma-1} n^{1-\alpha-\gamma} g^{'}(e) \qquad w = (1-\alpha-\gamma)(zk)^{\alpha} g(e)^{\gamma} n^{-\alpha-\gamma}$$

which imply:

$$n = \left[\frac{(1-\alpha-\gamma)(zk)^{\alpha} g(e)^{\gamma}}{w}\right]^{\frac{1}{\alpha+\gamma}}$$

replacing:

$$\begin{aligned} \frac{u_e}{u_c}\frac{h^{'}(e)}{g^{'}(e)} &= (1-\tau_k)\gamma (zk)^{\frac{\alpha}{\alpha+\gamma}} g(e)^{\frac{-\alpha}{\alpha+\gamma}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha+\gamma}} \\ g(e) &= \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\alpha+\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} zk \end{aligned}$$

So we get the desired result if it so happens that $\frac{h^{'}(e)}{g^{'}(e)}$ is constant, say ψ with $h(e) = \psi e$ and $g(e) = e$. If that is the case we can write labor demand as:

$$n = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\gamma}{\alpha}} zk$$

and profits as:

$$\begin{aligned} \pi(z,k) &= (zk)^{\alpha} g(e)^{\gamma} n^{1-\alpha-\gamma} - wn - rk \\ &= \left[\underbrace{(\alpha+\gamma)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z - r}_{\pi^{\star}(z)}\right] k \end{aligned}$$

Both profits and effort are proportional to how much capital the entrepreneur uses. The entrepreneur will only demand capital and operate their firm if the (after-tax) profits net of the effort cost are positive, that is:

$$k \geq 0 \longleftrightarrow (1-\tau_k)\,\pi^{\star}(z) - \underbrace{\frac{u_e h^{'}(e)}{u_c}}_{\text{Shadow Price}}\,\varepsilon(z) \geq 0,$$

where the shadow price of the effort cost is equal to ψ given our assumptions and

$$\varepsilon(z) \equiv \frac{e(z,k)}{k} = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\alpha+\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z.$$

In order to demand capital the entrepreneur must make profits to cover the cost of effort.

The optimal demand for capital is then:

$$k^{\star}(z,a) = \begin{cases} \lambda a & \text{if } \alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z > r \\ [0,\lambda a] & \text{if } \alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z = r \\ 0 & \text{if } \alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z < r \end{cases}$$

With this demand for capital we can replace back and get the level of profits, effort and labor demand.

Before proceeding to the optimal savings choice of the agent we need to determine the level of the capital demand for each type of entrepreneur. The relevant case has high-productivity entrepreneurs demanding $k^{\star}(z_h,a) = \lambda a$ for a total demand of $K_h = \lambda A_h$. The remaining assets are used by the low-productivity entrepreneurs who will be indifferent between any production level. The total demand for capital required to clear the market is $K_L = A_L - (\lambda-1)\,A_h$. Let $\lambda_{l,\iota} \equiv \frac{k_\iota}{a_\iota}$ be the ratio of capital to assets of low-productivity entrepreneur ι, for $\iota \in [0,1]$. We will show that the savings choice of the entrepreneur is independent of the value of $\lambda_{l,\iota}$.

Now we turn to the value function:

$$V_\iota(a,z) = \max_{\{c,a'\}} \ln(c - h(e_\iota)) + \beta E\left[V_\iota\left(a^{'},z^{'}\right)|z\right]$$
$$V_\iota(a,z) = \max_{\{c,a'\}} \ln(c - \psi e_\iota(z,a)) + \beta E\left[V_\iota\left(a^{'},z^{'}\right)|z\right]$$

Subject to:

$$c + a^{'} = R_\iota(z)\,a$$

where $R(z) \equiv (1-\tau_a) + (1-\tau_k)\,(r + \pi^{\star}(z)\,\lambda_\iota(z))$, and where $e_\iota(z,a) = \varepsilon(z)\,\lambda_\iota(z)\,a$. The value of $\lambda_\iota(z)$ satisfies:

$$\lambda_\iota(z) = \begin{cases} \lambda & \text{if } z = z_h \\ \lambda_{\iota,l} & \text{if } z = z_l. \end{cases}$$

We solve the dynamic programming problem of the entrepreneur via guess and verify. To this end, we guess that the value function of an entrepreneur of type $i \in \{l, h\}$ has the form

$$V_{i,\iota}(a) = m_{i,\iota} + n \log(a),$$

where $\{m_{l,\iota}, m_{h,\iota}\}_{\iota \in \{0,1\}}, n \in \mathbb{R}$ are coefficients. Under this guess the optimal savings choice of the entrepreneur is characterized by

$$\frac{1}{(R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota})\, a - a_i^{'}} = \frac{\beta n}{a_i^{'}}.$$

Solving for savings gives:

$$a_i^{'} = \frac{\beta n}{1+\beta n} (R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota})\, a.$$

Replacing the savings rule into the value function gives:

$$\begin{aligned} V_{i,\iota}(a) &= \log\left((R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota})\, a - a_i^{'}\right) + \beta\left(pV_{i,\iota}\left(a_i^{'}\right) + (1-p)\, V_{j,\iota}\left(a_i^{'}\right)\right) \\ m_{i,\iota} + n\log(a) &= \log\left((R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota})\, a - a_i^{'}\right) + \beta\left(pm_{i,\iota} + (1-p)\, m_{j,\iota}\right) + \beta n \log\left(a_i^{'}\right) \\ m_i + n\log(a) &= \beta n \log(\beta n) + (1+\beta n)\log\left(\frac{R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota}}{1+\beta n}\right) + \beta\left(pm_{i,\iota} + (1-p)\, m_{j,\iota}\right) + (1+\beta n)\log(a) \end{aligned}$$

Matching coefficients:

$$\begin{aligned} n &= 1 + \beta n \\ m_{i,\iota} &= \beta n \log(\beta n) + (1+\beta n)\log\left(\frac{R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota}}{1+\beta n}\right) + \beta\left(pm_{i,\iota} + (1-p)\, m_{j,\iota}\right), \end{aligned}$$

where $j \neq i$. The solution to the first equation implies:

$$n = \frac{1}{1-\beta},$$

which in turn delivers the optimal saving decision of the entrepreneur:

$$a' = \beta\left(R_\iota(z) - \psi\varepsilon(z)\,\lambda_\iota(z)\right) a. \tag{100}$$

Finally, we solve for the remaining coefficients for the relevant case in which high-productivity entrepreneurs are all constrained and low-productivity entrepreneurs are indifferent between any level of production. In that case, it holds that:

$$\begin{aligned} R_\iota(z_l) - \psi\varepsilon(z_l)\,\lambda_\iota(z_l) &= (1-\tau_a) + (1-\tau_k)\, r + \left[(1-\tau_k)\,\pi^\star(z) - \psi\varepsilon(z_l)\right]\lambda_\iota(z_l) \\ &= (1-\tau_a) + (1-\tau_k)\, r \end{aligned}$$

which is independent of the identity of the entrepreneur. It also holds that

$$\begin{aligned} R_\iota(z_h) - \psi\varepsilon(z_h)\lambda_\iota(z_h) &= (1-\tau_a) + (1-\tau_k)(r + \pi^\star(z_h)\lambda) - \psi\varepsilon(z_h)\lambda \\ &= (1-\tau_a) + (1-\tau_k)\left((1-\lambda)r + \alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z\lambda\right), \end{aligned}$$

which is also independent of the identity of the entrepreneur. Consequently, we can write without loss:

$$R_\iota(z) - \psi\varepsilon(z)\lambda_\iota(z) = R(z) - \psi\varepsilon(z)\lambda \equiv \hat{R}(z)$$

Having established these results, we can solve for m_l and m_h from the system of linear equations:

$$m_i = \frac{\beta}{1-\beta}\log\left(\frac{\beta}{1-\beta}\right) + \frac{1}{1-\beta}\log\left((1-\beta)\hat{R}(z)\right) + \beta(pm_i + (1-p)m_j)$$

The solution is given by:

$$m_i = \frac{\log(1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2}\log(\beta) + \frac{(1-\beta p)\log\hat{R}(z) + \beta(1-p)\log\hat{R}(z)}{(1-\beta)^2(1-\beta(2p-1))}$$

C.2.2 Equilibrium and aggregation

In equilibrium the interest rate is such that the low-productivity entrepreneurs are indifferent between lending their assets or using them in their own firm. Lending the assets gives them a (before-tax) return of r, using them gives them $\pi^\star(z_l)$ but it also entails a utility cost because of effort, which we know from the previous results is proportional to assets, same as returns and profits. The agents will be indifferent if the (after-tax) profits net of effort costs are zero:

$$\begin{aligned} 0 &= (1-\tau_k)\pi^\star(z_l) - \underbrace{\frac{u_e h'(e)}{u_c}}_{\text{Shadow Price}}\varepsilon(z_l) \\ 0 &= (1-\tau_k)\pi^\star(z_l) - \psi\varepsilon(z_l) \end{aligned}$$

replacing for the optimal solution of the entrepreneur's problem:

$$r = \alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z_l \tag{101}$$

We can then exploit the linearity of the savings function to aggregate results:

Lemma 6. *In the heterogenous return equilibrium* ($(\lambda-1)A_h < A_l$)*, output, wages, interest rate,* *and gross returns on savings are:*

$$Y = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}} \tag{102}$$

$$E = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{1}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}} \tag{103}$$

$$w = (1-\alpha-\gamma)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{ZK}{L}\right)^{\frac{\alpha}{1-\gamma}} \tag{104}$$

$$r = \alpha\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_l \tag{105}$$

$$R_{l,\iota} = (1-\tau_a) + (1-\tau_k)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} (\alpha+\gamma\lambda_\iota)\, z_l \tag{106}$$

$$R_h = (1-\tau_a) + (1-\tau_k)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} (\alpha z_\lambda + \gamma\lambda z_h) \tag{107}$$

and

$$\hat{R}(z) = R(z) - \psi\varepsilon(z)\,\lambda = \begin{cases} (1-\tau_a) + (1-\tau_k)\,\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_l & \text{if } z = z_l \\ (1-\tau_a) + (1-\tau_k)\,\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\lambda & \text{if } z = z_h \end{cases} \tag{108}$$

Proof. We start by considering the labor market clearing condition, we get

$$\begin{aligned} n^\star(z_h, K_h) + n^\star(z_l, K_l) &= L \\ \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\gamma}{\alpha}} (z_h K_h + z_l K_l) &= L \\ \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\gamma}{\alpha}} Q &= L \\ (1-\alpha-\gamma)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{Q}{L}\right)^{\frac{\alpha}{1-\gamma}} &= w \end{aligned}$$

Turning to the total effort we get:

$$\begin{aligned}
\left(\frac{E}{Q}\right)^{\alpha} &= \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\alpha+\gamma}\left(\frac{1-\alpha-\gamma}{w}\right)^{1-\alpha-\gamma}\\
\left(\frac{E}{Q}\right)^{\alpha} &= \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\alpha}\left(\frac{1-\alpha-\gamma}{w}\right)^{-\alpha}\left(\frac{L}{Q}\right)^{\alpha}\\
\left(\frac{E}{L}\right) &= \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)\left(\frac{1-\alpha-\gamma}{w}\right)^{-1}\\
E &= \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{1}{1-\gamma}} Q^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}} \qquad (109)
\end{aligned}$$

replacing back and then applying the result to the interest rate we get the usual Cobb-Douglas expressions:

$$w = (1-\alpha-\gamma)\,\frac{Q^{\alpha}E^{\gamma}L^{1-\gamma-\alpha}}{L} \qquad (110)$$

$$r = \alpha\frac{Q^{\alpha}E^{\gamma}L^{1-\gamma-\alpha}}{Q}z_l \qquad (111)$$

We can go further by replacing E which itself depends on other aggregates:

$$r = \alpha\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_l \qquad (112)$$

$$w = (1-\alpha-\gamma)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{Q}{L}\right)^{\frac{\alpha}{1-\gamma}} \qquad (113)$$

These two expressions also let us rewrite the profit rate (of capital) of entrepreneurs:

$$\pi^{\star}(z) = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}\left(\alpha\,(z-z_l)+\gamma z\right) > 0 \qquad (114)$$

Notice that profits are always positive for both types of entrepreneurs.

We can then use the equilibrium profit rates of entrepreneurs to rewrite the gross returns of entrepreneurs:

$$\begin{aligned}
R(z) &= (1-\tau_a) + (1-\tau_k)\,(r+\pi^{\star}(z)\,\lambda)\\
&= (1-\tau_a) + (1-\tau_k)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}\left(\alpha\,(z_l+\lambda\,(z-z_l))+\gamma\lambda z\right)
\end{aligned}$$

we can express this as:

$$R(z) = \begin{cases} (1-\tau_a) + (1-\tau_k)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}(\alpha+\gamma\lambda)\,z_l & \text{if } z = z_l\\ (1-\tau_a) + (1-\tau_k)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}(\alpha z_{\lambda}+\gamma\lambda z_h) & \text{if } z = z_h \end{cases} \qquad (115)$$

We are loosely referring as λ to the ratio of capital to assets of the entrepreneur. This ratio can vary by entrepreneur for the low-productivity entrepreneurs.

The return net of effort cost is:

$$\hat{R}(z) = R(z) - \psi\varepsilon(z)\lambda = (1-\tau_a) + (1-\tau_k)\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}(\lambda z + (1-\lambda)z_l)$$

More explicitly:

$$\hat{R}(z) = \begin{cases} (1-\tau_a) + (1-\tau_k)\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_l & \text{if } z = z_l \\ (1-\tau_a) + (1-\tau_k)\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\lambda & \text{if } z = z_h \end{cases} \tag{116}$$

Finally we consider aggregate output, for this note that the ratio of labor to capital is constant across entrepreneurs which allows us to aggregate in terms of the total capital of each type. We can express the output of an individual entrepreneur with productivity z and capital k as:

$$y(z,k) = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} zk$$

Aggregate output is the sum of the total output produced by each type of entrepreneur:

$$\begin{aligned} Y &= \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}}(z_h K_h + z_l K_l) \\ &= \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}\left(1-\frac{1-\alpha-\gamma}{1-\gamma}\right)}\left(\frac{Q}{L}\right)^{-\frac{1-\alpha-\gamma}{1-\gamma}} Q \\ &= \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}(ZK)^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}} \end{aligned}$$

For completeness we also consider the aggregate effort of high- and low-productivity entrepreneurs:

$$E_i \equiv \int e(z, k_{\iota,i})\, d\iota = \varepsilon(z_i)\int k_{\iota,i} d\iota = \left[\frac{(1-\tau_k)\gamma}{\psi} Q^{-(1-\alpha-\gamma)} L^{1-\alpha-\gamma}\right]^{\frac{1}{1-\gamma}} z_i K_i$$

This completes the derivation of the results.

□

We now turn to the evolution of aggregates:Using the savings decision rules of each type, we can obtain the law of motions for aggregate wealth held by each type as

$$A'_i = p\beta\hat{R}_i A_i + (1-p)\beta\hat{R}_j A_j. \tag{117}$$

Then the law of motion for aggregate wealth/capital ($K \equiv A_l + A_h$) becomes

$$K^{'} \;=\; \beta\left[\left(1-\tau_a\right)K + \left(1-\tau_k\right)\alpha\left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(ZK\right)^{\frac{\alpha}{1-\gamma}}\left(\frac{L}{Q}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}\right]. \qquad (118)$$

C.2.3 Steady state and changes in taxes

In steady state it must be that:

$$\frac{1}{\beta} = \left(1-\tau_a\right) + \left(1-\tau_k\right)\alpha\left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} Z^{\frac{\alpha}{1-\gamma}}\left(\frac{L}{K}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} \qquad (119)$$

We can use this to simplify the returns net of effort cost:

$$\hat{R}\left(z\right) = \begin{cases} \left(1-\tau_a\right) + \left(\frac{1}{\beta} - \left(1-\tau_a\right)\right)\frac{z_l}{Z} & \text{if } z = z_l \\ \left(1-\tau_a\right) + \left(\frac{1}{\beta} - \left(1-\tau_a\right)\right)\frac{z_\lambda}{Z} & \text{if } z = z_h \end{cases} \qquad (120)$$

which is the same as in Section 2.

Finally, from the individual law of motions of aggregate assets it must be that:

$$\frac{1-p\beta\hat{R}_h}{\left(1-p\right)\beta\hat{R}_l} = \frac{1-s_h}{s_h} = \frac{\left(1-p\right)\beta\hat{R}_h}{1-p\beta\hat{R}_l}$$

After some algebra, this implies:

$$1 - p\beta\left[\hat{R}_l + \hat{R}_h\right] + \left(2p-1\right)\beta^2\hat{R}_l\hat{R}_h = 0$$

we can further express this condition in terms of Z by replacing $R\left(z\right) - \psi\varepsilon\left(z\right)$:

$$\left(1-\left(1-\tau_a\right)\beta\left(2p-1\right)\right)Z^2 - \left(p-\left(1-\tau_a\right)\beta\left(2p-1\right)\right)\left(z_l+z_\lambda\right)Z + \left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)z_l z_\lambda = 0$$

which is the same expression for steady productivity as in Section 2.

Consequently, Propositions 1 and 2 apply to this economy without modifications:

Proposition 6. *Propositions 1 and 2 apply to this economy, so that a steady state equilibrium with heterogeneous returns exists if and only if $\lambda < \overline{\lambda}$, and an increase in wealth taxes in such an equilibrium increases productivity Z if and only if $p > 1/2$.*

The difference between the model in Section 2 and the model with effort is in the response of aggregate variables other than Z to changes in taxes. It turns out that all directions are maintained, but there are now two sources of changes on aggregates. The first source is, as in Section 2, a change in productivity. The second source is a direct effect of taxes on the effort of entrepreneurs. An increase in wealth taxes reduces capital income taxes which in turn reduces the distortions on the effort choice of entrepreneurs.

Before establishing the effects of a change in taxes on aggregate variables we revisit the role of government spending. The Government's constraint can be expressed just as before:

$$G = \tau_k \alpha Y + \tau_a K.$$

Assumption 1 still implies that:

$$\frac{1-\tau_k}{1-\beta\left(1-\tau_a\right)} = \frac{1-\theta}{1-\beta}$$

Then, steady state capital is, under Assumption 1:

$$K = \left(\frac{\alpha\beta\left(1-\theta\right)}{1-\beta}\right)^{\frac{1-\gamma}{1-\alpha-\gamma}} \left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\frac{\gamma}{1-\alpha-\gamma}} Z^{\frac{\alpha}{1-\alpha-\gamma}} L \tag{121}$$

Note that the level of capital depends directly on capital income taxes through their effect on effort. Alternatively, we can write the value of capital in terms of the level of wealth taxes:

$$K = \left(\alpha\beta\right)^{\frac{1-\gamma}{1-\alpha-\gamma}} \left(\frac{1-\theta}{1-\beta}\right)^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\left(1-\beta\left(1-\tau_a\right)\right)\gamma}{\psi}\right)^{\frac{\gamma}{1-\alpha-\gamma}} Z^{\frac{\alpha}{1-\alpha-\gamma}} L \tag{122}$$

This makes it clear that aggregate capital increases with wealth taxes both through the efficiency gains (higher Z) and the decrease in distortions, lower τ_k.

Lemma 7. *If $\lambda < \overline{\lambda}$ and $p > 1/2$ and after an increase in wealth taxes, the wealth share of high-productivity entrepreneurs increases, $\frac{ds_h}{d\tau_a} > 0$, the after-tax return net of effort costs of high-productivity entrepreneurs also increases, $\frac{d\hat{R}_h}{d\tau_a} > 0$, while the after-tax returns net of effort costs of low-productivity entrepreneurs decreases, $\frac{d\hat{R}_l}{d\tau_a} < 0$. The aggregate capital stock and increases, $\frac{dK}{d\tau_a} > 0$, as do total effort, output, and wages, $\frac{dE}{d\tau_a}, \frac{dY}{d\tau_a}, \frac{dw}{d\tau_a} > 0$.*

Proof. The wealth share of high-productivity entrepreneurs is tied to productivity by:

$$s_h = \frac{Z - z_l}{z_\lambda - z_l}$$

so that the wealth share changes in the same direction as productivity. Productivity increases following Proposition 2.

The results for after tax returns net of effort costs follow from a straightforward modification of Lemma 2 which gives:

$$\frac{d\left(R_h - \psi\varepsilon_h\right)}{d\tau_a} > 0 \qquad \text{and} \qquad \frac{d\left(R_l - \psi\varepsilon_l\right)}{d\tau_a} < 0$$

Total capital increases with wealth taxes:

$$\frac{d\log K}{d\log \tau_a} = \frac{\gamma}{1-\alpha-\gamma}\frac{\beta\tau_a}{1-\beta\left(1-\tau_a\right)} + \frac{\alpha}{1-\alpha-\gamma}\frac{d\log Z}{d\log \tau_a} > 0$$

It follows immediately that output, wages, and total effort increase since they depend positively on $Q = ZK$ and negatively on capital income taxes τ_k.

□

C.2.4 Welfare and optimal taxes

Introducing an effort choice for entrepreneurs changes the choice of optimal taxes in two direct ways. First, the equilibrium level of wages, and hence workers' welfare, depend on taxes directly through the effect of taxes on effort. Second, entrepreneurial welfare depends now on after-tax returns net of effort cost. However, only the first effect has an effect on the choice of optimal taxes. This is because in steady state the after-tax returns net of effort cost behave exactly like after-tax returns did in the model of Section 2. This leads to the following result:

Proposition 7. *The optimal wealth tax with entrepreneurial effort is higher than the optimal tax found in Proposition 4. Moreover, the optimal tax satisfies:*

$$n_w\frac{\gamma}{1-\alpha-\gamma}\left(\frac{\frac{\beta\tau_a}{1-\beta(1-\tau_a)}}{\frac{d\log Z}{d\log\tau_a}} + \frac{\alpha}{1-\alpha}\right) = -\left(n_w\frac{\alpha}{1-\alpha} + \frac{1-n_w}{2}\left(\xi_{\hat{R}_l}+\xi_{\hat{R}_h}\right)\right)$$

where $\xi_{\hat{R}_l}$ and $\xi_{\hat{R}_h}$ are equivalent to the elasticities of after-tax returns with respect to productivity in the model without entrepreneurial effort (Section 2).

Proof. The relevant welfare measures are:

$$\log\left(1+\text{CE}_{1,i}\right) \quad = \quad \begin{cases} \log {}^{w_a}/_{w_k} & \text{if } i = w \\ \frac{(1-\beta)\log {}^{\hat{R}_{a,i}}/_{\hat{R}_{k,i}} + \beta(1-p)\left(\log {}^{\hat{R}_{a,l}}/_{\hat{R}_{k,l}} + \log {}^{\hat{R}_{a,h}}/_{\hat{R}_{k,h}}\right)}{(1-\beta)(1-\beta(2p-1))} & \text{if } i \in \{l,h\} \end{cases} \tag{123}$$

and optimal taxes are set to maximize

$$\log\left(1+\text{CE}_1\right) = \sum_i n_i \log\left(1+\text{CE}_{1,i}\right).$$

The optimal is characterized by the following equation:

$$\frac{d\log\left(1+CE_1\right)}{d\tau_a} = 0$$

$$n_w\frac{d\log w}{d\tau_a} + \frac{1-n_w}{2}\left(\xi_{\hat{R}_l}+\xi_{\hat{R}_h}\right)\frac{d\log Z}{d\tau_a} = 0$$

where $\xi_{R_i-\psi\epsilon_i}$ are found just as in Lemma 2. The wage satisfies:

$$\frac{d\log w}{d\tau_a}=\frac{\gamma}{1-\alpha-\gamma}\frac{\beta}{1-\beta\left(1-\tau_a\right)}+\frac{\alpha}{1-\alpha-\gamma}\frac{d\log Z}{d\tau_a}$$

we can replace back to get:

$$n_w\left(\frac{\gamma}{1-\alpha-\gamma}\frac{\beta\tau_a}{1-\beta\left(1-\tau_a\right)}+\xi_w\frac{d\log Z}{d\log\tau_a}\right)=-\frac{1-n_w}{2}\left(\xi_{\hat{R}_l}+\xi_{\hat{R}_h}\right)\frac{d\log Z}{d\log\tau_a}$$

We do not have a closed form expression for the elasticity of productivity (Z) with respect to wealth taxes (τ_a), but this expression gives the optimal taxes.

We can also express the condition as:

$$n_w\frac{\gamma}{1-\alpha-\gamma}\left(\frac{\frac{\beta\tau_a}{1-\beta(1-\tau_a)}}{\frac{d\log Z}{d\log\tau_a}}+\frac{\alpha}{1-\alpha}\right)=-\left(n_w\frac{\alpha}{1-\alpha}+\frac{1-n_w}{2}\left(\xi_{\hat{R}_l}+\xi_{\hat{R}_h}\right)\right)$$

The right hand side is increasing in wealth taxes and it is in fact identical to the result in Proposition 4, while the left hand side is always positive. This leads to the conclusion that wealth taxes are higher with effort. The level of optimal taxes in Proposition 4 makes the right hand side of the equation zero, but the left hand side is still positive, so taxes must be higher.

$\square$

C.3 Excess Return

Proposition 8. ***(Existence and Uniqueness of Steady State)*** *There exists a unique steady state that features heterogenous returns if and only if*

$$\lambda < \overline{\lambda} \equiv 1 + \frac{(1-p) - \omega_h (2p-1)(1-\beta(1-\tau_a))}{p - (2p-1)\left(\beta(1-\tau_a) + (1-\beta(1-\tau_a))\frac{z_l}{z_h}\left(1-\omega_h\left(\frac{z_h}{z_l}-1\right)\right)\right)}$$

and either $\tau_a < \frac{1}{\beta}\left(\frac{1-p}{2p-1}\frac{1}{\omega_h} - (1-\beta)\right)$ *and* $\omega_h > 0$*, or* $\tau_a > \frac{1}{\beta}\left(\frac{1-p}{2p-1}\frac{1}{\omega_h} - (1-\beta)\right)$ *and* $\omega_h < 0$*.*

Proof. We evaluate

$$\begin{aligned} h(x) = {} & (1-(1-\tau_a)\beta(2p-1))x^2 - (z_\lambda + z_l)(p - \beta(1-\tau_a)(2p-1))x \\ & + (2p-1) z_l z_\lambda (1-\beta(1-\tau_a))\left(1+\omega_h - \omega_h \frac{x}{z_l}\right) \end{aligned}$$

at $x = z_l$ and $x = z_\lambda$ as before since $Z = s_h z_\lambda + (1-s_h) z_l$. For $x = z_l$ we obtain:

$$h(z_l) = (1-p) z_l (z_l - z_\lambda) < 0$$

and for $x = z_\lambda$ we obtain:

$$\begin{aligned} h(z_\lambda) &= (1-p) z_\lambda (z_\lambda - z_l) - (2p-1) z_l z_\lambda (1-\beta(1-\tau_a))\left(\frac{z_\lambda}{z_l} - 1\right)\omega_h \\ &= z_\lambda (z_\lambda - z_l)((1-p) - (2p-1)(1-\beta(1-\tau_a))\omega_h) \end{aligned}$$

Thus, there is a unique steady equilibrium Z iff $h(z_\lambda) > 0$, that is

$$\begin{aligned} 0 &< (1-p) - (2p-1)(1-\beta(1-\tau_a))\omega_h \\ 0 &< (1-p) - (2p-1)(1-\beta+\beta\tau_a)\omega_h \\ (2p-1)\beta\omega_h\tau_a &< (1-p) - (2p-1)(1-\beta)\omega_h \end{aligned}$$

There are two cases of interest:

$$\begin{aligned} \tau_a &< \frac{1-p}{(2p-1)\beta\omega_h} - \frac{(1-\beta)}{\beta} \text{ if } \omega_h > 0 \\ \tau_a &> \frac{1-p}{(2p-1)\beta\omega_h} - \frac{(1-\beta)}{\beta} \text{ if } \omega_h < 0. \end{aligned}$$

For returns to be heterogeneous before taxes and wedges we need that $Z < z_h$. To get a bound for this to be the case we evaluate:

$$\begin{aligned}
h\left(z_h\right) &= 0 \\
\left(1-\left(1-\tau_a\right)\beta\left(2p-1\right)\right)z_h^2-\left(z_\lambda+z_l\right)\left(p-\beta\left(1-\tau_a\right)\left(2p-1\right)\right)z_h & \\
+\left(2p-1\right)z_l z_\lambda\left(1-\beta\left(1-\tau_a\right)\right)\left(1+\omega_h-\omega_h\frac{z_h}{z_l}\right) &= 0 \\
\left(1-\left(1-\tau_a\right)\beta\left(2p-1\right)\right)\frac{z_h^2}{z_l^2}-\left(\frac{z_h}{z_l}+1\right)\left(p-\beta\left(1-\tau_a\right)\left(2p-1\right)\right)\frac{z_h}{z_l} & \\
+\left(2p-1\right)\frac{z_h}{z_l}\left(1-\beta\left(1-\tau_a\right)\right)\left(1+\omega_h-\omega_h\frac{z_h}{z_l}\right) & \\
-\left(\lambda-1\right)\left(\frac{z_h}{z_l}-1\right)\left[\left(p-\beta\left(1-\tau_a\right)\left(2p-1\right)\right)\frac{z_h}{z_l}+\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\left(1+\omega_h-\omega_h\frac{z_h}{z_l}\right)\right] &= 0 \\
\frac{z_h}{z_l}\left[\left(1-p\right)-\omega_h\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\right] & \\
-\left(\lambda-1\right)\frac{z_h}{z_l}\left[\left(p-\beta\left(1-\tau_a\right)\left(2p-1\right)\right)+\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_l}{z_h}\left(1-\omega_h\left(\frac{z_h}{z_l}-1\right)\right)\right] &= 0 \\
\left(1-p\right)-\omega_h\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right) & \\
-\left(\lambda-1\right)\left[p-\left(2p-1\right)\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_l}{z_h}\left(1-\omega_h\left(\frac{z_h}{z_l}-1\right)\right)\right)\right] &= 0
\end{aligned}$$

So the threshold is:

$$\overline{\lambda}<1+\frac{\left(1-p\right)-\omega_h\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)}{p-\left(2p-1\right)\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_l}{z_h}\left(1-\omega_h\left(\frac{z_h}{z_l}-1\right)\right)\right)}$$

which is (greater? lower?) than the one we had before:

$$\begin{aligned}
\frac{\partial\log\lambda}{\partial\omega_h} &= -\frac{\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)}{\left(1-p\right)-\omega_h\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)} \\
&\quad -\frac{\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_l}{z_h}\left(\frac{z_h}{z_l}-1\right)}{p-\left(2p-1\right)\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_l}{z_h}\left(1-\omega_h\left(\frac{z_h}{z_l}-1\right)\right)\right)} \\
&= -\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\left[\frac{1}{\left(1-p\right)-\omega_h\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)}\right. \\
&\quad \left.+\frac{\frac{z_l}{z_h}\left(\frac{z_h}{z_l}-1\right)}{p-\left(2p-1\right)\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_l}{z_h}\left(1-\omega_h\left(\frac{z_h}{z_l}-1\right)\right)\right)}\right]
\end{aligned}$$

We can evaluate this derivative at $\omega_h=0$:

$$\frac{\partial\log\lambda}{\partial\omega_h}\left(0\right)=-\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\left[\frac{1}{1-p}+\frac{\frac{z_l}{z_h}\left(\frac{z_h}{z_l}-1\right)}{p-\left(2p-1\right)\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_l}{z_h}\right)}\right]<0$$

So an increase in ω_h makes the bound more stringent, which makes sense as it increases the effective return of high-productivity entrepreneurs. A decrease of ω_h does the opposite.

$\square$

Lemma 8. *If $z_h = z_l = z$, then the steady state $Z = z$ and an increase in wealth tax increases the after-tax return R_i iff $\omega_{-i} < 0 < \omega_i$ but it does not increase TFP. Thus, an increase in the wealth tax does not increase the utility of workers and decreases the total utility of entrepreneurs. Overall, the optimal policy is to tax capital $\tau_k^\star > 0$ and subsidize wealth $\tau_a^\star < 0$.*

Proposition 9. ***(Efficiency Gains from Wealth Taxation)*** *If $\lambda < \overline{\lambda}$, aggregate productivity increases with wealth taxes, $\frac{dZ}{d\tau_a} > 0$, if individual productivity shocks are positively correlated, $p > 1/2$ and the after-tax return of high-productivity entrepreneurs is higher than the after-tax returns of low-productivity entrepreneurs:*

$$(1+\omega_l)\, z_l < (1+\omega_h)\, z_\lambda,$$

or productivity shocks are negatively correlated $p < 1/2$ and $(1+\omega_l)\, z_l > (1+\omega_h)\, z_\lambda$.

Corollary 5. *The wedges (ω_l, ω_h) satisfy $(1+\omega_l)\, z_l < (1+\omega_h)\, z_\lambda$ equilibrium if and only if :*

$$\omega_h > \underline{\omega}_h = -\frac{1}{2}\left(\frac{1-p}{1+p}\right)\left(\frac{z_\lambda - z_l}{z_\lambda}\right).$$

Proof. In order to see the effect of a higher wealth tax, take derivative of $h(x)$ with respect to τ_a:

$$\begin{aligned}
\frac{1}{\beta(2p-1)}\frac{dh(x)}{d\tau_a} =& x^2 - (z_\lambda + z_l)\, x + z_l z_\lambda \left(1 + \omega_h - \omega_h \frac{x}{z_l}\right) \\
=& x^2 - (z_\lambda + z_l)\, x + z_l z_\lambda + z_l z_\lambda \left(1 - \frac{x}{z_l}\right)\omega_h \\
=& (x - z_l)(x - z_\lambda) + z_\lambda (z_l - x)\, \omega_h \\
=& (x - z_l)(x - (1+\omega_h)\, z_\lambda)
\end{aligned}$$

Since $Z = s_h z_\lambda + (1-s_h)\, z_l$, we know that $Z - z_l > 0$ and, under $p > 1/2$, $\frac{dh(x)}{d\tau_a} < 0$ iff $Z - (1+\omega_h)\, z_\lambda < 0$. If high-productivity entrepreneurs are earning excess return $\omega_h > 0$, then $Z - (1+\omega_h)\, z_\lambda < 0$ and a higher wealth tax increases efficiency. If however, $\omega_h < 0$ then a higher wealth tax increases productivity iff $Z - (1+\omega_h)\, z_\lambda < 0$. Substituting $Z = s_h z_\lambda + (1-s_h)\, z_l$, we obtain

$$\begin{aligned}
s_h z_\lambda + (1-s_h)\, z_l - (1+\omega_h)\, z_\lambda &> 0 \\
(1-s_h)(z_l - z_\lambda) - \omega_h z_\lambda &> 0 \\
\omega_h &> -\frac{(1-s_h)(z_\lambda - z_l)}{z_\lambda}.
\end{aligned}$$

We will show that in equilibrium$\omega_h > -\frac{(1-s_h)(z_\lambda - z_l)}{z_\lambda}$ if and only if $(1+\omega_h)\, z_\lambda > (1+\omega_l)\, z_l$. First, we evaluate $\omega_l z_l A_l + \omega_h z_\lambda A_h = 0$, which implies

$$\omega_l z_l = -\omega_h z_\lambda \frac{s_h}{1-s_h}.$$

Then, we can then replace into $(1+\omega_h)\, z_\lambda > (1+\omega_l)\, z_l$ to obtain the result:

$$\begin{aligned} -\omega_h z_\lambda - \omega_h z_\lambda \frac{s_h}{1-s_h} &< z_\lambda - z_l \\ -\omega_h z_\lambda \left(\frac{1}{1-s_h}\right) &< z_\lambda - z_l \\ \omega_h &> -\left(1-s_h\right) \frac{z_\lambda - z_l}{z_\lambda}. \end{aligned}$$

So we have that if $p > 1/2$:

$$\frac{dZ}{d\tau_a} > 0 \longleftrightarrow Z < (1+\omega_h)\, z_\lambda \longleftrightarrow \omega_h > -\left(1-s_h\right) \frac{z_\lambda - z_l}{z_\lambda} \longleftrightarrow (1+\omega_h)\, z_\lambda > (1+\omega_l)\, z_l \longleftrightarrow R_h > R_l$$

Finally, we can check conditions that guarantee that $Z < (1+\omega_h)\, z_\lambda$ in terms of parameters, for this we evaluate $h\left((1+\omega_h)\, z_\lambda\right) = 0$ which, after some algebra, results in:

$$\omega_h > \underline{\omega}_h = -\frac{1}{2}\left(\frac{1-p}{1+p}\right)\left(\frac{z_\lambda - z_l}{z_\lambda}\right).$$

□

C.4 Stationary wealth distribution

The model presented in Section 2 as well as the extensions presented above do not have a stationary wealth distribution. Here we consider an alternative version of the model in which entrepreneurs have a permanent productivity type but are subject to mortality risk. In particular assume that entrepreneurs die with a constant probability $1-\delta$, upon death they are replaced by a new entrepreneur with initial assets $\overline{a}$ and whose productivity is z_i $(i \in \{h, l\})$ with probability $1/2$. The value of $\overline{a}$ is determined endogenously in equilibrium as the average bequest in the economy (which coincides with the average wealth). With respect to the main model of Section 2, this model loses the variation in productivity.[18] In exchange, this alternative version of the model exhibits a stationary wealth distribution that allows to better study how changes in taxes affect wealth inequality and welfare.

C.4.1 Entrepreneur's problem

The problem of an entrepreneur is now

$$V(a,z) = \max_{a'} \log(c) + \beta\delta V\left(a', z\right) \quad (124)$$
$$\text{s.t. } c + a' = R(z)\, a,$$

where $R(z)$ takes the same form as in Section 2.1. The solution takes the form $V_i(a) = m_i + n\log(a)$, where $n = \frac{1}{1-\beta\delta}$ and $m_i = \frac{1}{(1-\beta\delta)^2}\left[\beta\delta\log\beta\delta + (1-\beta\delta)\log(1-\beta\delta) + \log R_i\right]$, and implies an optimal savings rule

$$a' = \beta\delta R(z)\, a, \quad (125)$$

C.4.2 Evolution of aggregates and steady state

The savings choices of agents are still linear in assets, which lets us express the evolution of aggregate wealth as:

$$A_i^{'} = \beta\delta^2 R_i A_i + (1-\delta)\,\overline{a}, \quad (126)$$

so that in steady state:

$$A_i = \frac{1-\delta}{1-\beta\delta^2 R_i}\overline{a}, \quad (127)$$

where $\overline{a} \equiv K/2 = (A_l + A_h)/2$. We later show that $\beta\delta R_l < 1 < \beta\delta R_h < 1/\delta$, so that in equilibrium low types dissave and high types save, but not at rate that prevents the existence of a stationary equilibrium for the economy.

From the evolution of the low- and high-type assets we get:

$$K^{'} = \beta\delta^2\left(R_l A_l + R_h A_h\right) + (1-\delta)\,K$$

[18] It is straightforward to keep the variation in productivities across generations by redefining p as the probability that an individual entrepreneur keeps the productivity of the previous generation and setting two initial values of assets $\overline{a}_l$ and $\overline{a}_h$ for entrepreneurs born with productivity z_l and z_h respectively. The values of $\overline{a}_l$ and $\overline{a}_h$ are determined endogenously in equilibrium as the average bequest of each group of entrepreneurs, which are functions of the average wealth of low- and high-productivity entrepreneurs in steady state. We opt to abstract from this to keep the presentation of this new model as simple as possible.

replacing by the equilibrium value of returns $\left(R_i = (1-\tau_a) + (1-\tau_k)\,\alpha\,(ZK/L)^{\alpha-1} z_i\right)$ and evaluating in steady state we get:

$$(1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha}\left(\frac{K}{L}\right)^{\alpha-1} = \frac{1}{\beta\delta}, \tag{128}$$

which characterizes the level of steady state capital K given productivity, just as in (21).[19] Finally, joining the steady state conditions for low- and high- productivity entrepreneurs in equation (127) gives us a condition that characterizes the equilibrium of the model in terms of returns:

$$1 = \frac{(1-\delta)\left(1-\beta\delta^2\left(\frac{R_l+R_h}{2}\right)\right)}{(1-\beta\delta^2 R_l)(1-\beta\delta^2 R_h)}. \tag{129}$$

We can express this condition in terms of steady state productivity Z using (128) to get:

$$\left(1-\beta\delta^2(1-\tau_a)\right)Z^2 - (1+\delta(1-2\beta\delta(1-\tau_a)))\left(\frac{z_\lambda+z_l}{2}\right)Z + \delta(1-\beta\delta(1-\tau_a))\,z_l z_\lambda = 0 \tag{130}$$

which is again a quadratic equation in Z, as (23) in Section 2.3. The solution to this equation determines the steady state of the economy as well as the upper bound on the collateral constraint parameter λ that ensures that the economy is in the heterogeneous returns equilibrium:

Proposition 10. ***(Existence and Uniqueness of Steady State)*** *There exists a unique steady state. The steady state equilibrium features heterogenous returns* ($R_h > R_l$) *if and only if* $\lambda < \lambda_p^{\star} \equiv 1 + \frac{1-\delta}{1-\delta+2\delta(1-\beta\delta(1-\tau_a))\left(1-\frac{z_l}{z_h}\right)}$*. Moreover,* $\beta\delta R_l < 1 < \beta\delta R_h < 1/\delta$ *in steady state.*

Proof. We proceed in three steps. First, we determine conditions on the steady state value of Z that guarantee that $\beta\delta R_l < 1 < \beta\delta R_h < 1/\delta$. Second, we verify that there exists a solution to equation (130) satisfying those conditions. Finally, we prove that there is a unique root of (130) satisfying those conditions.

We start by showing that $R_l < 1/\beta\delta < R_h$. We verify this directly using equation (128) and the fact that $z_l < Z < z_\lambda$:

$$R_l = (1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha}\left(\frac{K}{L}\right)^{\alpha-1}\frac{z_l}{Z} < (1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha}\left(\frac{K}{L}\right)^{\alpha-1} = \frac{1}{\beta\delta}$$

and

$$\frac{1}{\beta\delta} = (1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha}\left(\frac{K}{L}\right)^{\alpha-1} < (1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha}\left(\frac{K}{L}\right)^{\alpha-1}\frac{z_\lambda}{Z} = R_h$$

[19] After imposing the steady state condition for capital, the after-tax rates of return become $R_i = 1-\tau_a + \frac{1-\beta\delta(1-\tau_a)}{\beta\delta}\frac{z_i}{Z}$.

Letting $\eta = \beta\delta(1-\tau_a)$, we can also show that $\beta\delta R_h < 1/\delta$ if $\frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda < Z$. Thus,

$$\beta\delta R_l < 1 < \beta\delta R_h < {}^1\!/\!_\delta \longleftrightarrow Z \in \left(\max\left\{z_l, \frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda\right\}, z_\lambda\right) \tag{131}$$

Note that the interval for Z is non-empty. This is immediate because:

$$z_l < z_\lambda \qquad \text{and} \qquad \frac{\delta(1-\eta)}{1-\delta\eta} < 1.$$

Moreover, the lower bound depends on the ratio of productivities. We have $\max\left\{z_l, \frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda\right\} = z_l$ if and only if $\frac{\delta(1-\eta)}{1-\delta\eta} \leq \frac{z_l}{z_\lambda}$.

Next, we show there exists a unique solution to equation (130) in the interval of equation (131). For this define

$$H(x) = (1-\delta\eta) - (1-\delta(2\eta-1)) \frac{\left(\frac{z_\lambda+z_l}{2}\right)}{x} + \delta(1-\eta)\frac{z_l z_\lambda}{x^2}$$

as the residual of equation (130) at x. We verify directly that H has a root in the interval $\left(\max\left\{z_l, \frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda\right\}, z_\lambda\right)$:

$$\begin{aligned} H(z_l) &= -\frac{1-\delta}{z_l}[z_\lambda - z_l] < 0 \\ H\left(\frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda\right) &= -\frac{1-\delta\eta}{2\delta(1-\eta)}\frac{1-\delta}{z_\lambda}[z_\lambda - z_l] < 0 \\ H(z_\lambda) &= \frac{1-\delta}{2z_\lambda}[z_\lambda - z_l] > 0 \end{aligned}$$

The existence of the unique root is guaranteed by the intermediate value theorem and the fact that the function is quadratic.

Now we derive sufficient conditions for the economy to be in the equilibrium with with excess supply of funds: $(A_l > (\lambda-1) A_h)$. This happens if and only if $Z \leq z_h$. So now we find conditions that guarantee that $H(z_h) > 0$ which implies that $Z \leq z_h$ since $H(Z) = 0$ and $H(z)$ is increasing in in $z \geq Z$.

$$H(z_h) = (1-\delta\eta) - (1-\delta(2\eta-1)) \frac{\left(\frac{z_\lambda+z_l}{2}\right)}{z_h} + \delta(1-\eta)\frac{z_l z_\lambda}{z_h^2} > 0$$

which after some manipulation gives:

$$\lambda < \overline{\lambda} \equiv 1 + \frac{1-\delta}{1-\delta+2\delta(1-\eta)\left(1-\frac{z_l}{z_h}\right)}$$

□

Just as in Section 3, we show that an increase in wealth taxes raises steady state productivity Z. Note that Z always increases with τ_a, that is because productivity is persistent by construction.

We will maintain the assumption that $\lambda < \lambda_p^\star$ in all the results that follow.

Proposition 11. ***(Efficiency Gains from Wealth Taxation)*** *If the steady state equilibrium features return heterogeneity* $\left(\lambda < \lambda_p^\star\right)$*, productivity increases with wealth taxes,* $\frac{dZ}{d\tau_a} > 0$.

Proof. We first define the auxiliary function

$$H\left(x;\tau_a\right) = \left(1 - \beta\delta^2\left(1-\tau_a\right)\right) - \left(1 + \delta\left(1 - 2\beta\delta\left(1-\tau_a\right)\right)\right)\frac{\left(\frac{z_\lambda + z_l}{2}\right)}{x} + \delta\left(1 - \beta\delta\left(1-\tau_a\right)\right)\frac{z_l z_\lambda}{x^2}$$

which characterizes the steady state if and only if $\lambda < \overline{\lambda}$. Simple manipulation of the function gives:

$$H\left(x;\tau_a\right) = 1 - \left[\left(1 + \frac{1}{\delta} - \frac{z_\lambda}{x}\right) z_l + \left(1 + \frac{1}{\delta} - \frac{z_l}{x}\right) z_\lambda\right]\frac{\delta}{2x} - \left(1 - \frac{z_l}{x}\right)\left(1 - \frac{z_\lambda}{x}\right)\beta\delta^2\left(1-\tau_a\right)$$

This function is decreasing in τ_a for $x \in \left(z_l, z_\lambda\right)$, which is the interval of the steady state value of Z similar to the one in Figure 3:

$$\frac{d\tilde{H}\left(x,\tau_a\right)}{d\tau_a} = \underbrace{\left(1 - \frac{z_l}{x}\right)}_{(+)}\underbrace{\left(1 - \frac{z_\lambda}{x}\right)}_{(-)}\beta\delta^2.$$

This gives the desired result: $\frac{dZ}{d\tau_a} > 0$.

□

The response of aggregate variables to changes in equilibrium Z (and hence to τ_a) follow the same patterns as in Section 3. In Appendix C.4.5 we present the equivalent results to Lemmas 3 and 3 describing the response of all aggregate variables in steady state.

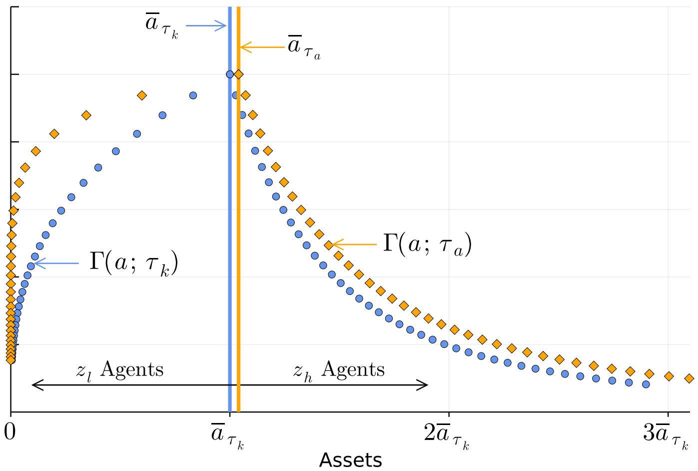
C.4.3 Stationary distribution of assets

We now derive the stationary distribution of assets. Note that all entrepreneurs are born with the same level of wealth $\overline{a}$ and then save at a constant rate during their lifetimes. In particular, high-types save at a (gross) rate $\beta\delta R_h > 1$ and low-types dissave at a (gross) rate $\beta\delta R_l < 1$. So, in the stationary equilibrium the wealth distribution of high-types has support in the interval $[\overline{a}, \infty)$ and the distribution of low-types in the interval $(0, \overline{a}]$. Moreover, the distribution of wealth is discrete, with endogenous mass points at $\left\{\overline{a}, \beta\delta R_h\overline{a}, \left(\beta\delta R_h\right)^2\overline{a}, \ldots\right\}$ for the high-types and $\left\{\overline{a}, \beta\delta R_l\overline{a}, \left(\beta\delta R_l\right)^2\overline{a}, \ldots\right\}$ for the low-types.

The share of entrepreneurs of type i with wealth $a = \left(\beta\delta R_i\right)^t\overline{a}$ is given by the share of agents who have lived exactly t periods:

$$\Gamma_i\left(\left(\beta\delta R_i\right)^t\overline{a}\right) = \Pr\left(\text{age} = t\right) = \delta^t\left(1-\delta\right) \tag{132}$$

Figure C.1: Stationary Distribution of Assets



Note: *The figure reports the stationary distribution of assets for two economies. The blue circles correspond to an economy with only capital income taxes ($\tau_k = \theta$ and $\tau_a = 0$) and its values are labeled with τ_k. The orange diamonds correspond to an economy with wealth taxes ($\tau_a = 10\%$ and τ_k set to satisfy Assumption 1) and its values are labeled with τ_a. The horizontal axis is presented in units of average assets in the capital income tax economy ($\overline{a}_{\tau_k}$). In both economies we set the remaining parameters as follows: $\beta = 0.96$, $\delta = 1 - 1/80$, $z_l = 1/2$, $z_h = 3/2$, $\theta = 25\%$, and $\lambda = 1.2$.*

So the distribution of wealth is a geometric distribution with parameter δ.[20]

Figure C.1 illustrates the behavior of the stationary distribution of assets. Agents are born with initial wealth $\overline{a}$ and save or dissave at constant rates depending on their productivity. A change in taxes affects the location of the mass-points of the distribution. In the figure, we contrast and economy without wealth taxes (that we label as τ_k) with one with wealth taxes (that we label as τ_a). The wealth tax economy has a higher level of overall wealth and hence $\overline{a}_{\tau_a}$ is to the right of $\overline{a}_{\tau_k}$. The change in $\overline{a}$ impacts all mass points (which are proportional to $\overline{a}$), shifting them rightwards. Then the increase in the dispersion of wealth is explained by the increase in the dispersion of returns, something reminiscent of the results in Lemma 3 and that we verify below for this economy.

Finally, we define a convenient measure of wealth concentration in the economy. Since wealth is determined by type and age, we can define the top wealth share as the fraction of wealth held by high types above an age t. This would correspond to the wealth share of the top $100 \times (1-\delta) \sum_{s=t}^{\infty} \delta^s = 100 \times \delta^t$ percent. Their total wealth is given by

$$A_{h,t} \equiv (1-\delta) \sum_{s=t}^{\infty} \left(\beta\delta^2 R_h\right)^s \overline{a} = \left(\beta\delta^2 R_h\right)^t A_h.$$

Then the top wealth shares are

$$s_{h,t} \equiv \frac{\left(\beta\delta^2 R_h\right)^t A_h}{K} = \left(\beta\delta^2 R_h\right)^t s_h. \tag{133}$$

[20]The characterization of the stationary distribution of assets mimics the derivations in Jones (2015) adapted to the discrete time setting.

After an increase in wealth taxes the dispersion of returns increases, this affects the distribution by shifting the mass points, although it does not affect the mass associated with each point, as shown in Figure C.1. Because s_h and R_h increase with the wealth tax, the top wealth share $s_{h,t} = \left(\beta\delta^2 R_h\right)^t s_h$ increase with the wealth tax.

Lemma 9. ***(Top-Wealth Shares and Wealth Taxes)*** *An increase in wealth taxes increases the top-wealth-shares (133). The percentage increase in the wealth share is higher for higher wealth levels.*

Proof. The result is immediate from the definition of wealth shares as a function of after-tax returns (equation 133) and the fact that R_h increases with wealth taxes (see Lemma 12 in Appendix C.4). An increase in wealth taxes increases the returns of high-productivity entrepreneurs (R_h), which in turn increases their savings rate and asset holdings. The effect is compounded with age because savings rate are constant, increasing more the wealth holdings of older/wealthier entrepreneurs.

□

C.4.4 Welfare and optimal taxes

We focus on the average welfare gain taking advantage of the characterization of the stationary wealth distribution. This leads to a welfare measure that is closely related to the CE_2 measure presented in equation (36). However, just as in Section 3.2 we can also define an individual welfare measure equivalent to the CE_1 measure defined in equation (34). Workers' welfare behaves just as in Section 3.2, but the welfare measures of low- and high-productivity entrepreneurs now depend only on their own returns. Hence, $\text{CE}_{1,h}$ is always positive and $\text{CE}_{1,l}$ is always negative. Total entrepreneurial welfare still depends on (log-)average returns (which decrease with wealth taxes) and optimal taxes are characterized similarly to Proposition 4. We provide details for these results in Appendix C.4.

We compute the average welfare gain by each type (denoting it as $\text{CE}_{2,i}$) in the following way

$$\sum_a \left(V_k(a,i) + \frac{\log(1+\text{CE}_{2,i})}{1-\beta\delta}\right)\Gamma_k(a,i) \quad = \quad \sum_a V_a(a,i)\,\Gamma_a(a,i)\,.$$

This average measure depends on the assets of each agent through their distribution, and thus captures the effects of higher capital accumulation triggered by the tax reform. Using the age distribution, we obtain:

$$\log(1+\text{CE}_{2,i}) = \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)}\log\frac{R_{a,i}}{R_{k,i}} + \log\frac{K_a}{K_k}. \tag{134}$$

The utilitarian welfare gain for the whole population CE_2 is given by $\log(1+\text{CE}_2) = \sum_i n_i \log(1+\text{CE}_{2,i})$. For entrepreneurs, $\text{CE}_{2,i}$ welfare gains depend on the accumulation of capital in the economy. Interestingly, the effect of aggregate capital is the same for both types of entrepreneurs. This is because they both benefit from starting their lives at a higher level of initial assets (recall that $\overline{a} = K/2$) and their future asset levels are all proportional to their initial wealth (as discussed in Section C.4.3). This makes it possible even for low-productivity entrepreneurs to benefit from the increase in wealth taxes if elasticity of output with respect to capital is sufficiently high. We summarize these results in Lemma 7.

Lemma 10. ***(Welfare Gain by Agent Type)*** *In the perpetual youth model, for high-productivity entrepreneurs, $CE_{2,h} > CE_{1,h} > 0$ always. For low-productivity entrepreneurs $CE_{1,l} < 0$ always and $CE_{2,l} > 0$ for a marginal increase in wealth taxes if and only if*

$$\xi_K \geq \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)}\xi_{R_l}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z.

Proof. From the definition of $\mathrm{CE}_{2,i}$ in (134) we get:

$$\log(1+\mathrm{CE}_{2,i}) = \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)}\log\frac{R_{a,i}}{R_{k,i}} + \log\frac{K_a}{K_k}$$

It is immediate that $\mathrm{CE}_{2,h} > 0$ because K and R_h are both increasing in wealth taxes (Lemmas 11 and 12). Moreover, $\mathrm{CE}_{2,h} - \mathrm{CE}_{1,h} = \log {}^{K_a}/_{K_k} > 0$ because K is increasing in wealth taxes.

For $\mathrm{CE}_{2,l}$ consider the derivative of the welfare measure with respect to wealth taxes:

$$\begin{aligned}
\frac{d\log(1+\mathrm{CE}_{2,l})}{d\tau_a} &= \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)}\frac{1}{R_l}\frac{R_l}{d\tau_a} + \frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a} \\
&= \left[-\frac{1-\beta\delta^2}{2\beta\delta^2(1-\beta\delta)}\frac{1}{R_l}\frac{z_\lambda - z_l}{(z_\lambda - Z)^2}Z + \frac{\alpha}{1-\alpha}\right]\frac{1}{Z}\frac{dZ}{d\tau_a} \\
&= \left[-\frac{1-\beta\delta^2}{(1-\beta\delta)}\frac{(1-s_h)}{(2(1-s_h)-(1-\delta))}\frac{z_\lambda - z_l}{(z_\lambda - Z)^2}Z + \frac{\alpha}{1-\alpha}\right]\frac{1}{Z}\frac{dZ}{d\tau_a} \\
&= \left[-\frac{(1-\beta\delta^2)(z_\lambda - z_l)Z}{(1-\beta\delta)(2(1-s_h)-(1-\delta))(1-s_h)} + \frac{\alpha}{1-\alpha}\right]\frac{1}{Z}\frac{dZ}{d\tau_a}
\end{aligned}$$

This defines bounds on α above which there are welfare gains for the low-productivity entrepreneurs.

□

We next study the optimal tax problem of the government using the CE_2 measure. Taking the changes in the wealth distribution into account changes the optimal combination of taxes, but, as in Section (3.2), does not change the key tradeoffs at play. However, unlike in Section (3.2), taking wealth accumulation into account does not necessarily lead to higher optimal wealth taxes or lower $\underline{\alpha}$ and $\overline{\alpha}$ thresholds. Higher initial wealth increases the benefits from the reform (we again have $\xi_w = \xi_K = {}^{\alpha}/_{1-\alpha}$), but also increases the losses from the lower expected returns due to the compounding effect of returns on individual asset accumulation (which are suffered by the low-productivity entrepreneurs). In proposition (10) we characterize the optimal tax levels that maximizes CE_2.

Proposition 12. ***(Optimal CE$_2$ Taxes)*** *In the perpetual youth model, the optimal tax combination $\left(\tau_{a,2}^{*},\tau_{k,2}^{*}\right)$ that maximizes the utilitarian welfare measure CE$_2$ is unique and given by the solution to the following equation:*

$$n_w\xi_w+(1-n_w)\,\xi_K=\xi=-\,(1-n_w)\,\frac{1-\beta\delta^2}{(1-\delta)\,(1-\beta\delta)}\left(\frac{\xi_{R_l}+\xi_{R_h}}{2}\right) \tag{135}$$

where $\xi_x\equiv\frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Furthermore, there exist two cutoff values for α which we denote $\underline{\alpha}_2$ and $\overline{\alpha}_2$ such that $\left(\tau_{a,2}^{\star},\tau_{k,2}^{\star}\right)$ satisfies the following properties:

$$\begin{aligned}
&\tau_{a,2}^{\star}\in\left[1-\frac{1}{\beta\delta},0\right) \text{ and } \tau_{k,2}^{\star}>\theta && \text{if } \alpha<\underline{\alpha}_2\\
&\tau_{a,2}^{\star}\in\left[0,\frac{\theta\,(1-\beta\delta)}{\beta\delta\,(1-\theta)}\right] \text{ and } \tau_{k,2}^{\star}\in[0,\theta] && \text{if } \underline{\alpha}_2\le\alpha\le\bar{\alpha}_2\\
&\tau_{a,2}^{\star}\in\left(\frac{\theta\,(1-\beta\delta)}{\beta\delta\,(1-\theta)},\tau_{a,2}^{\max}\right) \text{ and } \tau_{k,2}^{\star}<0 && \text{if } \alpha>\bar{\alpha}_2
\end{aligned}$$

where $\tau_{a,2}^{\max}\ge 1$, $\underline{\alpha}_2$ and $\overline{\alpha}_2$ are the solutions to equation (135) with $\tau_a=0$ and $\tau_a=\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$, respectively. Recall from Lemma (3) that $\xi={}^{\alpha}/_{1-\alpha}$.

Proof. For aggregate welfare:

$$\begin{aligned}
\frac{\log\left(1+\mathrm{CE}_2\right)}{(1-\beta\delta)}&=n_w\left(V_a(w)-V_k(w)\right)+\sum_{i\in\{l,h\}}n_i\left(\sum_a V_a(a,i)\,\Gamma_a(a,i)-\sum_a V_k(a,i)\,\Gamma_k(a,i)\right)\\
\frac{\log\left(1+\mathrm{CE}_2\right)}{(1-\beta\delta)}&=n_w\left(\frac{\log\left(1+\mathrm{CE}_{2,w}\right)}{1-\beta\delta}\right)+\sum_{i\in\{l,h\}}n_i\frac{\log\left(1+\mathrm{CE}_{2,i}\right)}{1-\beta\delta}\\
\log\left(1+\mathrm{CE}_2\right)&=\sum_{i\in\{w,l,h\}}n_i\log\left(1+\mathrm{CE}_{2,i}\right)
\end{aligned}$$

where $\mathrm{CE}_{2,w}=\mathrm{CE}_{1,w}$. The optimal wealth tax is characterized by:

$$\begin{aligned}
\frac{d\log\left(1+\mathrm{CE}_2\right)}{d\tau_a}&=0\\
\left[\frac{d\log\left(1+\mathrm{CE}_2\right)}{d\log Z}\right]\frac{d\log Z}{d\tau_a}&=0\\
\left[n_w\frac{d\log w}{d\log Z}+\frac{1-n_w}{2}\frac{1-\beta\delta^2}{(1-\delta)\,(1-\beta\delta)}\left(\frac{d\log R_l}{d\log Z}+\frac{d\log R_h}{d\log Z}\right)+(1-n_w)\frac{d\log K}{d\log Z}\right]\frac{d\log Z}{d\tau_a}&=0\\
\left[n_w\xi_w+(1-n_w)\,\xi_K+(1-n_w)\,\frac{1-\beta\delta^2}{(1-\delta)\,(1-\beta\delta)}\left(\frac{\xi_{R_l}+\xi_{R_h}}{2}\right)\right]\frac{d\log Z}{d\tau_a}&=0
\end{aligned}$$

As in the proof of Proposition 4 the above condition is satisfied if and only if

$$n_w \xi_w + (1 - n_w) \xi_K = -(1 - n_w) \frac{1 - \beta\delta^2}{(1 - \delta)(1 - \beta\delta)} \left(\frac{\xi_{R_l} + \xi_{R_h}}{2} \right)$$

and from Lemma 3 we get $\xi_w = \xi_K = \xi = \alpha/1-\alpha$. The uniqueness of the solution and the definition of the thresholds for α and its implications for the optimal taxes follow from the same arguments as in Proposition 4. We can further replace to get a more explicit formula:

$$\frac{\alpha}{1-\alpha} + \frac{1-n_w}{2} \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)} \frac{Z}{R_h R_l} \frac{dR_h R_l}{dZ} = 0$$

$$\frac{\alpha}{1-\alpha} + \frac{1-n_w}{2} \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)} \frac{Z}{R_h R_l} \frac{\left(1-\delta^2\right)(1-2s_h)}{\left(2\beta\delta^2 s_h (1-s_h)\right)^2} \frac{ds_h}{dZ} = 0$$

$$\frac{\alpha}{1-\alpha} + \frac{1-n_w}{2} \frac{1-\beta\delta^2}{(1-\delta)(1-\beta\delta)} \frac{1}{R_h R_l} \frac{\left(1-\delta^2\right)(1-2s_h)}{\left(2\beta\delta^2 s_h (1-s_h)\right)^2} \frac{Z}{z_\lambda - z_l} = 0$$

Relative to the condition for CE_1 in Appendix C.4, the first term is now larger (multiplied by 1 instead of n_w), the second term is also larger (multiplied by a factor $\frac{1-\beta\delta^2}{1-\delta} > 1$).

□

Corollary 6. ***(Comparison of CE_1 and CE_2 Taxes)*** *In the perpetual youth model, optimal wealth taxes are higher when taking the wealth accumulation into account* $\left(\tau^\star_{a,2} > \tau^\star_a\right)$ *and the α-thresholds are lower ($\underline{\alpha}_2 < \underline{\alpha}$ and $\overline{\alpha}_2 < \overline{\alpha}$) if* $\frac{1-\beta\delta^2}{1-\delta} < \frac{1}{n_w}$.

C.4.5 Perpetual youth model: Additional results

Aggregate variables in steady state We will use the fact that under Assumption 1, the government budget constraint reduces to

$$\frac{1-\theta}{1-\beta\delta} = \frac{1-\tau_k}{1-\beta\delta(1-\tau_a)}. \tag{136}$$

Lemmas 11 and 12 parallel Lemmas 3 and 3:

Lemma 11. ***(Aggregate Variables in Steady State)*** *The steady state aggregate variables satisfy*

$$s_h = \frac{Z - z_l}{z_\lambda - z_l} > \frac{1}{2} \qquad \frac{ds_h}{dZ} > 0 \tag{137}$$

$$R_h = \frac{1}{\beta\delta^2}\left(1 - \frac{1-\delta}{2s_h}\right) \qquad \frac{dR_h}{dZ} > 0 \tag{138}$$

$$R_l = \frac{1}{\beta\delta^2}\left(1 - \frac{1-\delta}{2(1-s_h)}\right) \qquad \frac{dR_l}{dZ} < 0. \tag{139}$$

Moreover, under Assumption 1

$$K=\left(\alpha\frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}}Z^{\frac{\alpha}{1-\alpha}}L \qquad \frac{dK}{dZ}\propto\frac{\alpha}{1-\alpha}Z^{\frac{2\alpha-1}{1-\alpha}}<0 \tag{140}$$

$$Q=\left(\alpha\frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}Z\right)^{\frac{1}{1-\alpha}}L \qquad \frac{dQ}{dZ}\propto\frac{1}{1-\alpha}Z^{\frac{\alpha}{1-\alpha}}>0 \tag{141}$$

$$Y=\left(ZK\right)^{\alpha}L^{1-\alpha} \qquad \frac{dY}{dZ}\propto\frac{\alpha}{1-\alpha}Z^{\frac{2\alpha-1}{1-\alpha}}>0 \tag{142}$$

$$A_h=\frac{Z-z_l}{z_\lambda-z_l}K \qquad \frac{dA_h}{dZ}\propto\frac{Z^{\frac{2\alpha-1}{1-\alpha}}}{1-\alpha}\left(Z-\alpha z_l\right)>0 \tag{143}$$

$$A_l=\frac{z_\lambda-Z}{z_\lambda-z_l}K \qquad \frac{dA_l}{dZ}\propto\frac{Z^{\frac{\alpha}{1-\alpha}}}{1-\alpha}\left(\alpha z_\lambda-Z\right). \tag{144}$$

Proof. We know that $s_h=\frac{Z-z_l}{z_\lambda-z_l}$, so $s_h>1/2$ is equivalent to $Z>\frac{z_\lambda+z_l}{2}$. We can verify if this is the case by evaluating the residual of (130) at $\frac{z_\lambda+z_l}{2}$:

$$\begin{aligned}H\left(\frac{z_\lambda+z_l}{2}\right)&=\left(1-\delta\eta\right)-\left(1-\delta\left(2\eta-1\right)\right)+\delta\left(1-\eta\right)\frac{z_lz_\lambda}{\left(\frac{z_\lambda+z_l}{2}\right)^2}\\&=-\delta\left(1-\eta\right)+\delta\left(1-\eta\right)\frac{z_lz_\lambda}{\left(\frac{z_\lambda+z_l}{2}\right)^2}\\&=-\delta\left(1-\eta\right)\left(\frac{z_\lambda-z_l}{z_\lambda+z_l}\right)^2<0\end{aligned}$$

The residual is always negative. So it must be that $Z>\frac{z_\lambda+z_l}{2}$ and thus $s_h>1/2$.The result for the wealth share of the high-types is immediate from the definition of Z.

From the steady state level of wealth of high-productivity entrepreneurs we know that:

$$R_h=\frac{1}{\beta\delta^2}\left(1-\frac{1-\delta}{2s_h}\right)$$

which implies:

$$\frac{dR_h}{dZ}=\frac{1-\delta}{2\beta\delta^2}\frac{1}{s_h^2}\frac{ds_h}{dZ}>0$$

A similar calculation delivers:

$$R_l=\frac{1}{\beta\delta^2}\left(1-\frac{1-\delta}{2\left(1-s_h\right)}\right) \qquad \frac{dR_l}{dZ}=-\frac{1-\delta}{2\beta\delta^2}\frac{1}{\left(1-s_h\right)^2}\frac{ds_h}{dZ}<0$$

From (128) we can express government spending as:

$$G = \left(\tau_k + \tau_a \frac{\beta\delta(1-\tau_k)}{1-\beta\delta(1-\tau_a)}\right)\alpha Y,$$

and under Assumption 1 we get:

$$\frac{1-\theta}{1-\beta\delta} = \frac{1-\tau_k}{1-\beta\delta(1-\tau_a)}.$$

Replacing in (128) we get:

$$K = \left(\alpha \frac{\beta\delta(1-\theta)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} Z^{\frac{\alpha}{1-\alpha}} L$$

which is increasing in Z. From the same expression we get:

$$Q = ZK = \left(\alpha \frac{\beta\delta(1-\theta)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} Z^{\frac{1}{1-\alpha}} L$$

which is also increasing in Z. From this it is immediate that $Y = Q^{\alpha} L^{1-\alpha}$ is also increasing in Z.

Since K and s_h increase it must be the case that $A_h = s_h K$ increases as well. We are left with the response of A_l. To get it we first write A_l in terms of Z using the definition of the wealth share of the high-types:

$$\begin{aligned} A_l &= (1-s_h)A \\ &= \left(1 - \frac{Z - z_l}{z_\lambda - z_l}\right)A \\ &= \left(\alpha \frac{\beta\delta(1-\theta)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} L \frac{z_\lambda - Z}{z_\lambda - z_l} Z^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Taking derivatives shows that A_l decreases with Z (and hence with τ_a):

$$\frac{dA_l}{dZ} \propto \frac{Z^{\frac{\alpha}{1-\alpha}-1}}{z_\lambda - z_l}\left[\alpha z_\lambda - Z\right]$$

which is negative if $\alpha z_\lambda < Z$.

□

Returns can be expressed in terms of Z as before

$$R_l = 1 - \tau_a + \left(\frac{1}{\beta\delta} - (1-\tau_a)\right)\frac{z_l}{Z} \quad \text{and} \quad R_h = 1 - \tau_a + \left(\frac{1}{\beta\delta} - (1-\tau_a)\right)\frac{z_\lambda}{Z}$$

Similarly, the change in returns can be divided into the use-it-or-lose-it effect ($-(1 - {}^{z_l}/_{Z}) < 0$ and $-(1 - {}^{z_\lambda}/_{Z}) > 0$) and a negative general equilibrium effect.

Lemma 12. $\frac{dR_l}{d\tau_a} < 0$ *and* $\frac{dR_h}{d\tau_a} > 0$, $\frac{d(R_h + R_l)}{d\tau_a} < 0$ *and* $\frac{d(R_h R_l)}{d\tau_a} < 0$.

Proof. We know the share of wealth of the high-types is increasing along with the overall wealth in the economy, so A_h must increase as well, this will imply that R_h must have risen. From Lemma 11:

$$\frac{dR_h}{d\tau_a} = \frac{1-\delta}{2\beta\delta^2}\frac{1}{s_h^2}\frac{ds_h}{d\tau_a} = \frac{1-\delta}{2\beta\delta^2}\frac{z_\lambda - z_l}{(Z-z_l)^2}\frac{dZ}{d\tau_a} > 0,$$

and:

$$\frac{dR_l}{d\tau_a} = -\frac{1-\delta}{2\beta\delta^2}\frac{1}{(1-s_h)^2}\frac{ds_h}{d\tau_a} = \frac{-(1-\delta)}{2\beta\delta^2}\frac{z_\lambda - z_l}{(z_\lambda - Z)^2}\frac{dZ}{d\tau_a} < 0.$$

With this we get:

$$\frac{d(R_h+R_l)}{d\tau_a} = \frac{1-\delta}{2\beta\delta^2}\left(\frac{1-2s_h}{s_h^2(1-s_h)^2}\right)\frac{ds_h}{d\tau_a} = \frac{1-\delta}{2\beta\delta^2}\frac{(z_\lambda - z_l)^2(z_\lambda + z_l - 2Z)}{(Z-z_l)^2(z_\lambda - Z)^2}\frac{dZ}{d\tau_a}$$

so that $\frac{d(R_h+R_l)}{d\tau_a} \geq 0$ if and only if $s_h \leq 1/2$. Since $s_h > 1/2$ then $\frac{d(R_h+R_l)}{d\tau_a} < 0$.

Finally, we consider the product of returns, which is also decreasing in taxes.

$$\begin{aligned}
\frac{dR_hR_l}{d\tau_a} &= R_h\frac{dR_l}{d\tau_a} + R_l\frac{dR_h}{d\tau_a} \\
&= \frac{1-\delta}{2(\beta\delta^2)^2}\frac{ds_h}{d\tau_a}\left[-\left(1-\frac{1-\delta}{2s_h}\right)\frac{1}{(1-s_h)^2} + \left(1-\frac{1-\delta}{2(1-s_h)}\right)\frac{1}{s_h^2}\right] \\
&= \frac{1-\delta}{(2\beta\delta^2 s_h(1-s_h))^2}\frac{ds_h}{d\tau_a}\left[-s_h(2s_h-(1-\delta)) + (1-s_h)(2(1-s_h)-(1-\delta))\right] \\
&= \frac{1-\delta^2}{(2\beta\delta^2 s_h(1-s_h))^2}(1-2s_h)\frac{ds_h}{d\tau_a} \\
&= \frac{1-\delta^2}{(2\beta\delta^2(Z-z_l)(z_\lambda - Z))^2}(z_\lambda + z_l - 2Z)(z_\lambda - z_l)^2\frac{dZ}{d\tau_a} < 0.
\end{aligned}$$

This completes the proof.

□

Individual welfare comparisons Just as in Section 3.2 we can define the welfare change for each individual of type i asking how much they value being dropped from the capital income tax economy with $\tau_a = 0$ and $\tau_k = \theta$ to the economy with a positive wealth tax $\tau_a > 0$ in terms of lifetime consumption. We denote this consumption equivalent welfare measure as $\mathrm{CE}_1(a,i)$ and it is given by

$$\log(1+\mathrm{CE}_1(a,i)) = (1-\beta\delta)(V_a(a,i) - V_k(a,i)) = (1-\beta\delta)\Delta V(a,i). \tag{145}$$

All the terms containing wealth cancel, thus, the welfare gain depends only on the individual's type i. Consequently, we drop wealth "a" from the welfare measure below and write

$$\log(1+\mathrm{CE}_{1,i}) = \begin{cases} \log w_a/w_k & \text{if } i = w \\ \frac{1}{1-\beta\delta}\log R_{a,i}/R_{k,i} & \text{if } i \in \{l,h\}. \end{cases} \tag{146}$$

Workers' welfare increases with wealth taxes because of the effect of taxes on productivity Z and through it on wages (Lemma 11). From Lemma 12 we conclude that the welfare of high-productivity entrepreneurs goes up if wealth taxes increase, while the welfare of low-productivity entrepreneurs goes down. Recall that productivity types are permanent in this economy, because of that high-types do not take into account the effect of taxes on the returns of low-types, and vice versa. We summarize these results in Lemma 13.

Lemma 13. *$CE_{1,w} > 0$, $CE_{1,h} > 0$, and $CE_{1,l} < 0$.*

Proof. The result is immediate from (146) and Lemmas **??** and 12. As wealth taxes increase wages increase which increases the welfare of workers. The returns of high-productivity entrepreneurs increase, which increases their welfare. The returns of low-productivity entrepreneurs decrease, which decreases their welfare.

□

We next turn to aggregate welfare. As before the aggregate welfare measure can be computed as $\log(1 + \mathrm{CE}_1) = \sum_i n_i \log(1 + \mathrm{CE}_{1,i})$. Substituting in $\mathrm{CE}_{1,i}$'s gives

$$\log(1 + \mathrm{CE}_1) = n_w \frac{\alpha}{1-\alpha} \log(Z_a/Z_k) + \frac{1-n_w}{2(1-\beta\delta)} \left(\log {}^{R_{a,l}}/{}_{R_{k,l}} + \log {}^{R_{a,h}}/{}_{R_{k,h}}\right). \tag{147}$$

Again, this result parallels that of our benchmark model, and the same forces are in play when determining the optimal taxes. The total welfare of entrepreneurs decreases with wealth taxes because of lower average returns while workers' welfare increases because of higher wages. In proposition 13 we characterize the optimal tax problem of the government that maximizes the utilitarian welfare by choosing the optimal combination (τ_a, τ_k). The optimal tax combination equates the elasticity of wages with respect to productivity with the average elasticity of returns, weighted by population.

Proposition 13. *The optimal tax combination (τ_a^*, τ_k^*) that maximizes the utilitarian welfare measure CE_1 is unique and given by the solution to the following equation:*

$$n_w \xi_w = -\frac{1-n_w}{1-\beta\delta} \left(\frac{\xi_{R_l} + \xi_{R_h}}{2}\right) \tag{148}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Applying Lemma 11 gives

$$n_w \frac{\alpha}{1-\alpha} = \frac{1}{2} \frac{1-n_w}{1-\beta\delta} \frac{1}{R_h R_l} \frac{\left(1-\delta^2\right)(2s_h - 1)}{\left(2\beta\delta^2 s_h (1-s_h)\right)^2} \frac{Z}{z_\lambda - z_l}, \tag{149}$$

where τ_k is given from equation (136), $s_h = \frac{Z-z_l}{z_\lambda - z_l}$, Z is the solution to equation (23) and R_i's are given by equations (138) and (139). Furthermore, there exist two cutoff values for α which we *denote $\underline{\alpha}_p$ and $\overline{\alpha}_p$ such that $(\tau_a^\star, \tau_k^\star)$ satisfies the following properties:*

$$\tau_a^\star \in \left[1 - \frac{1}{\beta\delta}, 0\right) \textit{ and } \tau_k^\star > \theta \qquad \textit{if } \alpha < \underline{\alpha}_p$$
$$\tau_a^\star \in \left[0, \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}\right] \textit{ and } \tau_k^\star \in [0,\theta] \qquad \textit{if } \underline{\alpha}_p \leq \alpha \leq \bar{\alpha}_p$$
$$\tau_a^\star \in \left(\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}, \tau_{a,p}^{\max}\right) \textit{ and } \tau_k^\star < 0 \qquad \textit{if } \alpha > \bar{\alpha}_p$$

where $\tau_{a,p}^{\max} \geq 1$, $\underline{\alpha}_p$ is the solution to equation (149) with $\tau_a = 0$ and $\overline{\alpha}_p$ is the solution to equation (149) with $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$.

Proof. From (147) we get an expression for the total CE_1 welfare measure:

$$\sum_i n_i \log(1 + \mathrm{CE}_{1,i}) = n_w \frac{\alpha}{1-\alpha} \log(Z_a/Z_k) + \frac{1-n_w}{2(1-\beta\delta)} \log\left(\frac{R_{a,h}R_{a,l}}{R_{k,h}R_{k,l}}\right)$$

This is equal to 0 when $\tau_a = 0$ by construction. For welfare to increase we need the derivative of (147) with respect to τ_a to be positive. The derivative is:

$$n_w \frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a} + \frac{1}{2}\frac{1-n_w}{1-\beta\delta}\frac{1}{R_h R_l}\frac{dR_h R_l}{d\tau_a} > 0$$
$$n_w \frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a} + \frac{1}{2}\frac{1-n_w}{1-\beta\delta}\frac{1}{R_h R_l}\frac{\left(1-\delta^2\right)(1-2s_h)}{\left(2\beta\delta^2 s_h(1-s_h)\right)^2}\frac{ds_h}{d\tau_a} > 0$$
$$\left[n_w \frac{\alpha}{1-\alpha} + \frac{1}{2}\frac{1-n_w}{1-\beta\delta}\frac{1}{R_h R_l}\frac{\left(1-\delta^2\right)(1-2s_h)}{\left(2\beta\delta^2 s_h(1-s_h)\right)^2}\frac{Z}{z_\lambda - z_l}\right]\frac{1}{Z}\frac{dZ}{d\tau_a} > 0$$

where $s_h = \frac{Z-z_l}{z_\lambda - z_l}$, $1 - s_h = \frac{z_\lambda - Z}{z_\lambda - z_l}$, $R_l = \frac{1}{\beta\delta^2}\left(1 - \frac{1-\delta}{2(1-s_h)}\right)$, and and $R_h = \frac{1}{\beta\delta^2}\left(1 - \frac{1-\delta}{2s_h}\right)$. From Proposition 9 we know that $\frac{dZ}{d\tau_a} > 0$, so an increase in wealth taxes increases welfare if and only if

$$n_w \frac{\alpha}{1-\alpha} - \frac{1}{2}\frac{1-n_w}{1-\beta\delta}\frac{1}{R_h R_l}\frac{\left(1-\delta^2\right)(2s_h - 1)}{\left(2\beta\delta^2 s_h(1-s_h)\right)^2}\frac{Z}{z_\lambda - z_l} \geq 0.$$

Moreover, the optimal level of taxes is given by $\tau_a^\star$ for which this equation holds with equality.

□

The optimal tax combination has a positive wealth tax if $\alpha > \underline{\alpha}_p$ and the optimal wealth tax is greater than the tax reform level $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$ if $\alpha > \overline{\alpha}_p$, in that case the optimal capital income tax is a subsidy. The thresholds $\underline{\alpha}_p$ and $\bar{\alpha}_p$ can be solved explicitly in terms of parameters when $z_l = 0$. In this case $Z = \frac{(1+\delta)/2 - \beta\delta^2(1-\tau_a)}{1-\beta\delta^2(1-\tau_a)} z_\lambda$, $s_h = \frac{Z}{z_\lambda}$, $R_l = 1 - \tau_a$, and $R_h = \frac{1 - \frac{\delta\beta(1+\delta)(1-\tau_a)}{2}}{\delta\beta((1+\delta)/2 - \beta\delta^2(1-\tau_a))}$. Substituting these variables into the first order condition of the government and evaluating at $\tau_a = 0$ and $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$ gives the two α thresholds.

Corollary 7. $\frac{\underline{\alpha}_p}{1-\underline{\alpha}_p} = \frac{1-n_w}{2n_w}\frac{(1+\delta)\left(1-\beta\delta^2\right)^2}{(1-\delta)\beta\delta^2\left(1-\frac{\delta\beta(1+\delta)}{2}\right)}$ *and* $\frac{\bar{\alpha}_p}{1-\bar{\alpha}_p} = \frac{1-n_w}{2n_w}\frac{(1+\delta)\left(1-\theta-\beta\delta^2+\theta\delta\right)^2}{(1-\delta)(\beta\delta-\theta)\delta\left((1-\theta)-\frac{(1+\delta)(\beta\delta-\theta)}{2}\right)(1-\theta)}$ *if* $z_l = 0$.

Proof. We substitute the expressions $Z = \frac{(1+\delta)/2-\beta\delta^2(1-\tau_a)}{1-\beta\delta^2(1-\tau_a)}z_\lambda$, $s_h = \frac{Z}{z_\lambda} = \frac{(1+\delta)/2-\beta\delta^2(1-\tau_a)}{1-\beta\delta^2(1-\tau_a)}$, $1-s_h = \frac{(1-\delta)/2}{1-\beta\delta^2(1-\tau_a)}$, $2s_h - 1 = \frac{\delta-\beta\delta^2(1-\tau_a)}{1-\beta\delta^2(1-\tau_a)}$, $R_l = 1-\tau_a$, and $R_h = \frac{1-\frac{\delta\beta(1+\delta)(1-\tau_a)}{2}}{\delta\beta((1+\delta)/2-\beta\delta^2(1-\tau_a))}$ into

$$\begin{aligned}\frac{\alpha}{1-\alpha} &= \frac{1}{2}\frac{1-n_w}{(1-\beta\delta)\left(2\beta\delta^2\right)^2 n_w}\frac{1}{R_hR_l}\frac{\left(1-\delta^2\right)(2s_h-1)}{\left(s_h(1-s_h)\right)^2}\frac{Z}{z_\lambda}\\ &= \frac{1-n_w}{2n_w}\frac{(1+\delta)\left(1-\beta\delta(1-\tau_a)\right)\left(1-\beta\delta^2(1-\tau_a)\right)^2}{(1-\beta\delta)\beta\delta^2\left(1-\frac{\delta\beta(1+\delta)(1-\tau_a)}{2}\right)(1-\tau_a)(1-\delta)}.\end{aligned}$$

Evaluating this expression at $\tau_a = 0$ gives

$$\begin{aligned}\frac{\underline{\alpha}_p}{1-\underline{\alpha}_p} &= \frac{1-n_w}{2n_w}\frac{(1+\delta)\left(1-\beta\delta^2\right)^2}{(1-\delta)\beta\delta^2\left(1-\frac{\delta\beta(1+\delta)}{2}\right)}\\ \underline{\alpha}_p &= \frac{\frac{1-n_w}{2n_w}\frac{(1+\delta)\left(1-\beta\delta^2\right)^2}{(1-\delta)\beta\delta^2\left(1-\frac{\delta\beta(1+\delta)}{2}\right)}}{1+\frac{1-n_w}{2n_w}\frac{(1+\delta)(1-\beta\delta^2)^2}{(1-\delta)\beta\delta^2\left(1-\frac{\delta\beta(1+\delta)}{2}\right)}}.\\ \underline{\alpha}_p &= 1-\frac{1}{1+\frac{1-n_w}{2n_w}\frac{(1+\delta)(1-\beta\delta^2)^2}{(1-\delta)\beta\delta^2\left(1-\frac{\delta\beta(1+\delta)}{2}\right)}}.\end{aligned}$$

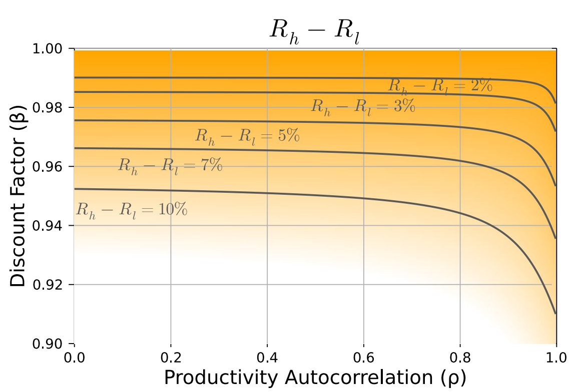
Evaluating at $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$ $\left(1-\tau_a = 1-\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)} = \frac{\beta\delta-\theta}{\beta\delta(1-\theta)}\right)$

$$\begin{aligned}\frac{\bar{\alpha}_p}{1-\bar{\alpha}_p} &= \frac{1-n_w}{2(1-\beta\delta)\beta\delta^2 n_w}\frac{(1+\delta)\left(1-\beta\delta\frac{\beta\delta-\theta}{\beta\delta(1-\theta)}\right)\left(1-\beta\delta^2\frac{\beta\delta-\theta}{\beta\delta(1-\theta)}\right)^2}{\left(1-\frac{\delta\beta(1+\delta)}{2}\frac{\beta\delta-\theta}{\beta\delta(1-\theta)}\right)\frac{\beta\delta-\theta}{\beta\delta(1-\theta)}(1-\delta)}\\ &= \frac{1-n_w}{2n_w}\frac{(1+\delta)\left(1-\theta-\beta\delta^2+\theta\delta\right)^2}{(1-\delta)(\beta\delta-\theta)\delta\left(1-\theta-\frac{(1+\delta)(\beta\delta-\theta)}{2}\right)(1-\theta)}\\ \bar{\alpha}_p &= 1-\frac{1}{1+\frac{1-n_w}{2n_w}\frac{(1+\delta)(1-\theta-\beta\delta^2+\theta\delta)^2}{(1-\delta)(\beta\delta-\theta)\delta\left((1-\theta)-\frac{(1+\delta)(\beta\delta-\theta)}{2}\right)(1-\theta)}}.\end{aligned}$$

□

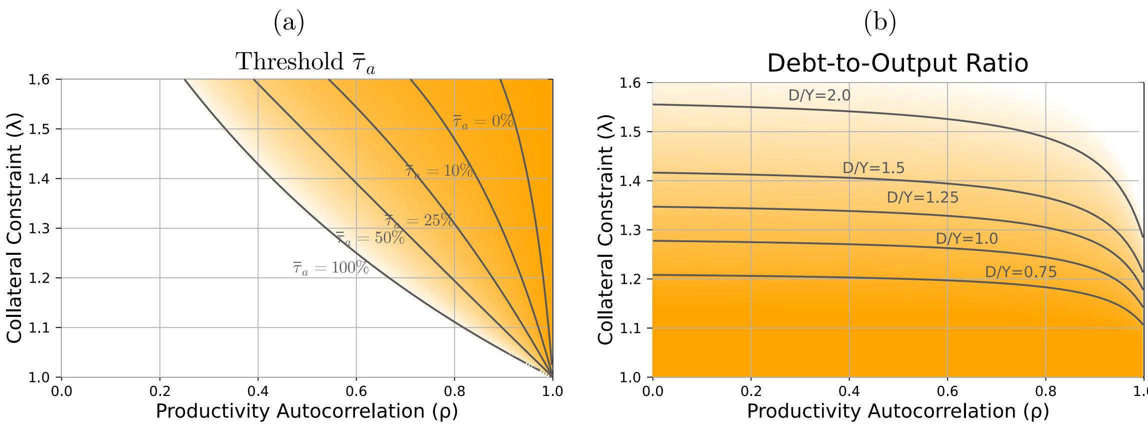
D Extra Tables and Figures

Figure D.2: Steady State Return Dispersion



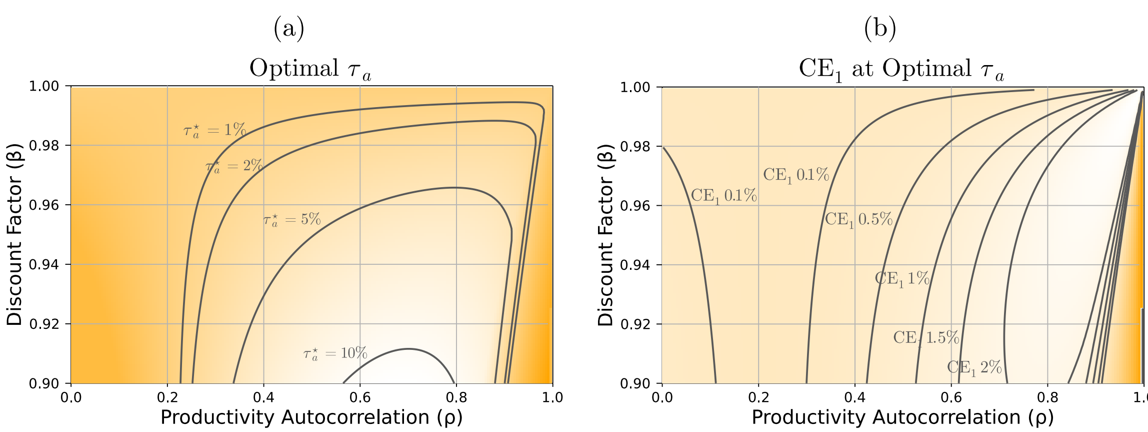
Note: *Figure D.2 reports the steady state return dispersion $R_h - R_l$ for combinations of the autocorrelation of productivity (ρ) and the discount factor (β). We set the remaining parameters as follows: $z_l = 0$, $z_h = 2$, $\tau_k = 25\%$, $\lambda = 1.32$, and $\alpha = 0.4$.*

Figure D.3: τ_a Conditions for Steady State with Heterogeneous Returns



Note: *Figure D.3a reports the value of $\overline{\tau}_a$ found in Corollary 1 for combinations of the autocorrelation of productivity (ρ) and the collateral constraint parameter (λ). The steady state exhibits heterogeneous returns if and only if $\tau_a \leq \overline{\tau}_a$. Figure D.3b reports the debt-to-output ratio when $\tau_a = 0$ computed as $(\lambda-1)A_h/Y$ for the same combinations of ρ and λ. In both figures we set the remaining parameters as follows: $z_l = 0$, $z_h = 2$, $\tau_k = 25\%$, $\beta = 0.96$, and $\alpha = 0.4$.*

Figure D.4: Optimal Wealth Taxes and Welfare Gain



Note: *Figure D.4a reports the value of the wealth taxes that maximize CE_1 welfare as described in Proposition 4 for combinations of the autocorrelation of productivity (ρ) and the discount factor (β). Figure D.4b reports the value of CE_1 welfare at the optimal wealth taxes. The value of τ_a is found by finding the root of equation (35). The value of τ_k satisfies equation (24). In both figures we set the remaining parameters as follows: $z_l = 0$, $z_h = 2$, $\theta = 25\%$, and $\lambda = 1.32$.*