

Linear Algebra — Practice Problems (Solutions)

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These practice questions accompany the Linear Algebra lecture and will be discussed in the review session.

Question 1

Compute the following matrix and vector products:

a. $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 1 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 5 \\ 1 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 7 & 1 \\ 0 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix}$

d. $\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

SOLUTIONS

Each entry of the product is the inner product of the corresponding row of the first matrix with the corresponding column of the second.

a.

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + 1 \cdot 1 & 3 \cdot 1 + 1 \cdot 5 & 3 \cdot 0 + 1 \cdot 2 \\ 2 \cdot 2 + 4 \cdot 1 & 2 \cdot 1 + 4 \cdot 5 & 2 \cdot 0 + 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 2 \\ 8 & 22 & 8 \end{bmatrix}$$

b.

$$\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 1 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 5 \\ 1 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 \cdot 2 + 0 \cdot 1 + 1 \cdot 4 & 0 \cdot 0 + 0 \cdot 3 + 1 \cdot 1 & 0 \cdot 5 + 0 \cdot 1 + 1 \cdot 2 \\ 3 \cdot 2 + 1 \cdot 1 + 1 \cdot 4 & 3 \cdot 0 + 1 \cdot 3 + 1 \cdot 1 & 3 \cdot 5 + 1 \cdot 1 + 1 \cdot 2 \\ -2 \cdot 2 + 3 \cdot 1 + 0 \cdot 4 & -2 \cdot 0 + 3 \cdot 3 + 0 \cdot 1 & -2 \cdot 5 + 3 \cdot 1 + 0 \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & 2 \\ 11 & 4 & 18 \\ -1 & 9 & -7 \end{bmatrix}$$

c.

$$\begin{bmatrix} 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 7 & 1 \\ 0 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 \cdot 2 + 3 \cdot 0 + 2 \cdot 2 & 5 \cdot 7 + 3 \cdot 4 + 2 \cdot 3 & 5 \cdot 1 + 3 \cdot 5 + 2 \cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 53 & 28 \end{bmatrix}$$

d.

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + 3 \cdot (-1) \\ 1 \cdot 2 + (-2) \cdot (-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Question 2

Being **formal and explicit** in the rules of matrix algebra (e.g. when things are commutative, distributive, when you require invertibility, etc.) solve the following equations for $x \in \mathbb{R}^N$, with constants: vector $b, c \in \mathbb{R}^N$, matrices A, D, Q, R all $\mathbb{R}^{N \times N}$, and scalar $m \in \mathbb{R}$,

- $xA = b + c$
- $Ax + (x^\top D)^\top = b$
- $Q^{-1}(Axm + mb) = Rx$

SOLUTIONS

- Starting from $xA = b + c$:
 - Assume A is invertible
 - $xA A^{-1} = (b + c)A^{-1}$, right multiplying by A^{-1}
 - $xI = (b + c)A^{-1}$, using $AA^{-1} = I$
 - $x = (b + c)A^{-1}$, using $Ix = x$
- Starting from $Ax + (x^\top D)^\top = b$:

- $Ax + D^\top(x^\top)^\top = b$, using $(AB)^\top = B^\top A^\top$
- $Ax + D^\top x = b$, using $(x^\top)^\top = x$
- $(A + D^\top)x = b$, using distributive property of left-multiplication $(A + B)x = Ax + Bx$
- If $A + D^\top$ is invertible, then can do additional steps:
- $(A + D^\top)^{-1}(A + D^\top)x = (A + D^\top)^{-1}b$, using left multiplication
- $x = (A + D^\top)^{-1}b$, using $A^{-1}A = I$ and $Ix = x$

c. Starting from $Q^{-1}(Axm + mb) = Rx$:

- $Q^{-1}(mAx + mb) = Rx$, using commutativity of scalar multiplication
- $Q^{-1}m(Ax + b) = Rx$, using distributivity of scalar multiplication
- $Q^{-1}(Ax + b) = \frac{1}{m}Rx$, using commutativity of scalar multiplication
- $QQ^{-1}(Ax + b) = Q\frac{1}{m}Rx$, left multiply by Q
- $Ax + b = \frac{1}{m}QRx$, using $QQ^{-1} = I$ and commutativity of scalar multiplication
- $b = \frac{1}{m}QRx - Ax$, rearranging sides
- $b = (\frac{1}{m}QR - A)x$, using distributive property of left-multiplication $(A+B)x = Ax + Bx$
- If $(\frac{1}{m}QR - A)$ is invertible, then can do additional steps:
- $x = (\frac{1}{m}QR - A)^{-1}b$, using left multiplication and then $A^{-1}A = I$

Question 3

Write the following as a system of equations in matrix form:

- $\begin{cases} 2x + 3y = 2 \\ x - 2y = -1 \end{cases}$ where $\{x, y\}$ are variables
- $\begin{cases} 2x - 3y = 1 \\ 3x + my = -2 \end{cases}$ where $\{x, y\}$ are variables
- $\begin{cases} 2a + b - 1 = 0 \\ 3b + 4c = 2 \\ -2a + c = 0 \end{cases}$ where $\{a, b, c\}$ are variables
- $\begin{cases} a + b = -3 \\ c - y - 4b = 0 \end{cases}$ where $\{a, b, c\}$ are variables

SOLUTIONS

- $\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
- $\begin{bmatrix} 2 & -3 \\ 3 & m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

- c. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, note moved the -1 to the right hand side
- d. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3 \\ y \end{bmatrix}$, moved the y to the right hand side since it is a constant
given stated assumptions

Question 4

- a. Find $G \in \mathbb{R}^2$ such that $G \cdot [a \ b]$ returns the second element, b .
- b. Find $H \in \mathbb{R}^{1 \times 2}$ such that $H [x \ y]^\top$ returns the sum $x + y$.
- c. Find a 2×2 matrix M such that $M [p \ q]^\top = [q \ p]^\top$ (i.e., swaps the two elements).
- d. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that doubles the first coordinate and leaves the second unchanged. Write a matrix representation of T .

SOLUTIONS

- a. $G = [0 \ 1]^\top$, with or without the transpose since uses a dot-product
- b. $H = [1 \ 1]$
- c. $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- d. $T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Question 5

- a. Find a vector $x \in \mathbb{R}^2$ such that $x \cdot [1 \ 2] = 0$
- b. Find a vector $x \in \mathbb{R}^2$ such that $x \cdot [0 \ 1] = 1$
- c. Find 3 vectors of unit length in $\mathbb{R}^3$ which are mutually orthogonal
- d. Given vectors x^1 and x^2 which may have a different norm (i.e., length), how can I use the norm and inner product to test if they are orthogonal? collinear?

SOLUTIONS

- a. $x = [2 \ -1]$
- b. $x = [0 \ 1]$
- c. The canonical orthonormal basis works: $x^1 = [1 \ 0 \ 0]$, $x^2 = [0 \ 1 \ 0]$, $x^3 = [0 \ 0 \ 1]$. Each has unit length and every pair has zero inner product. (Note: 3 is

the maximum — there is no set of 4 mutually orthogonal nonzero vectors in $\mathbb{R}^3$.)
d. If $x^1 \cdot x^2 = 0$ then they are orthogonal.

- To test if they are collinear, one approach might be to make them the same length with $\frac{x^1}{\|x^1\|_2} \cdot \frac{x^2}{\|x^2\|_2}$ and check whether the inner product $\frac{x^1}{\|x^1\|_2} \cdot \frac{x^2}{\|x^2\|_2} = \pm 1$

Question 6

- What is the span of the vectors $\begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}$?
- What is the span of the vectors $\begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \end{bmatrix}$?
- What is the span of the vectors $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$?
- For a set of vectors $x^1, \dots, x^k \in \mathbb{R}^n$, what is the highest dimension the span could be? The lowest?

SOLUTIONS

- All of $\mathbb{R}^2$
- All of $\mathbb{R}^2$. Note that $x^3 \equiv x^2 - x^1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Hence x^1 and x^3 can span anything as they are the canonical basis vectors.
- The set $\left\{ \alpha \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}^\top + \beta \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}^\top : \alpha, \beta \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top : x_3 = 2(x_1 + x_2) \right\}$, i.e. the plane $2x_1 + 2x_2 - x_3 = 0$ through the origin.
- The highest dimension it could be is $\min(k, n)$ (this requires the vectors to be linearly independent). The lowest it could be is 0, which happens if all x^i are the zero vector.

Question 7

Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Write the matrix-vector product Ab as a linear combination of the columns of the matrix.

SOLUTIONS

$$Ab = b_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Question 8

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \equiv [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

- Write the matrix-vector product Ab as a linear combination of the columns of A .
- What is the span of the columns of A ?
- Come up with 3 different variations on $[b_1 \ b_2 \ b_3]^\top$ that solve $Ab = [2 \ 3 \ 0]^\top$, or explain why it is not possible.
- Come up with 3 different variations on $[b_1 \ b_2 \ b_3]^\top$ that solve $Ab = [2 \ 3 \ 1]^\top$, or explain why it is not possible.

SOLUTIONS

a. $Ab = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

b. The span of the columns of A is all vectors of the form $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$, i.e., the xy -plane in

$\mathbb{R}^3$. Note that the third column is a linear combination of the first two columns:
 $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so the columns are not all linearly independent.

c. We need $b_1 + b_3 = 2$ and $b_2 + b_3 = 3$. Examples: $b \equiv [2 \ 3 \ 0]^\top$, $b \equiv [1 \ 2 \ 1]^\top$,
 $b \equiv [0 \ 1 \ 2]^\top$

- Key here: there is more than one way to do it because of collinearity in the columns of A —will be important for solving systems of equations and understanding identification
- Note that the span of the columns of A is a plane in $\mathbb{R}^3$

d. None exist because that vector is not in the span of the columns of A (its third entry is nonzero, but every column of A has a zero third entry)

Question 9

Let A and B be invertible $n \times n$ matrices. Show that $B^{-1}A^{-1}$ is the inverse of AB — that is, $(AB)^{-1} = B^{-1}A^{-1}$. (Note the order reversal.)

SOLUTIONS

Verify that $B^{-1}A^{-1}$ undoes AB from both sides:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I,$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Since $B^{-1}A^{-1}$ is both a left and a right inverse of the square matrix AB , AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. The factors reverse order: the matrix applied last is undone first. (We used associativity to regroup, and the assumed existence of A^{-1} and B^{-1} .)

PhD Track

Building on the PhD-track sections of the Linear Algebra lecture: projection onto the column space, the structure of $X'X$, and quadratic forms and definiteness.

PhD Track

Question 10

Let X be an $n \times k$ matrix with full column rank, and define the projection matrix $P = X(X'X)^{-1}X'$ and the annihilator $M = I - P$.

- Show that P is idempotent ($P^2 = P$) and symmetric ($P' = P$).
- Show that $MX = 0$, and interpret this geometrically.
- Show that $PM = 0$, and conclude that the fitted vector Py and the residual vector My are orthogonal for every y .

SOLUTIONS

Full column rank makes $X'X$ invertible, and $(X'X)^{-1}$ is symmetric because $X'X$ is.

a. **Idempotent:**

$$P^2 = X(X'X)^{-1} \underbrace{X'X(X'X)^{-1}}_I X' = X(X'X)^{-1}X' = P.$$

Symmetric: using $(ABC)' = C'B'A'$ and $((X'X)^{-1})' = (X'X)^{-1}$,

$$P' = (X(X'X)^{-1}X')' = X((X'X)^{-1})'X' = X(X'X)^{-1}X' = P.$$

- $MX = (I - P)X = X - X(X'X)^{-1}X'X = X - X = 0$. Geometrically, M annihilates every column of X , hence all of $\text{col}(X)$: it strips off the part of a vector lying in the column space and keeps only the orthogonal (residual) component.

c. $PM = P(I - P) = P - P^2 = P - P = 0$. Therefore, for any y ,

$$(Py)'(My) = y'P'My = y'PM y = y'(PM)y = 0,$$

using $P' = P$. The fitted values Py and residuals My are orthogonal — the geometric core of least squares.

PhD Track

Question 11

Let A be symmetric ($A = A'$) with eigenpairs $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$, where $\lambda_1 \neq \lambda_2$. Show that the eigenvectors are orthogonal: $v_1'v_2 = 0$.

SOLUTIONS

Evaluate the scalar $v_1'Av_2$ two ways.

Directly, $v_1'Av_2 = v_1'(\lambda_2 v_2) = \lambda_2 v_1'v_2$.

Using symmetry, $v_1'A = v_1'A' = (Av_1)' = (\lambda_1 v_1)' = \lambda_1 v_1'$, so $v_1'Av_2 = \lambda_1 v_1'v_2$.

Setting the two equal gives $(\lambda_1 - \lambda_2)v_1'v_2 = 0$. Since $\lambda_1 \neq \lambda_2$, it must be that $v_1'v_2 = 0$.

PhD Track

Question 12

Let P be a square matrix whose columns $v_1, \dots, v_n$ are orthonormal: $v_i'v_j = 0$ for $i \neq j$ and $v_i'v_i = 1$. Show that $P'P = I$, and conclude that $P^{-1} = P'$.

SOLUTIONS

The (i, j) entry of $P'P$ is the i -th row of P' (namely v_i') times the j -th column of P (namely v_j):

$$(P'P)_{ij} = v_i'v_j = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$$

which is exactly I . So $P'P = I$, i.e. P' is a left inverse of P .

The orthonormal columns are linearly independent: if $\sum_i \alpha_i v_i = 0$, taking the inner product with v_j gives $\alpha_j = 0$. So the square matrix P has independent columns and is invertible. Right-multiplying $P'P = I$ by P^{-1} yields $P' = P^{-1}$.

Question 13

Let A be symmetric positive definite with spectral decomposition $A = PDP'$, where P is orthogonal ($P'P = I$) and $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ with every $\lambda_i > 0$. Define $D^{1/2} = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$ and $K = D^{1/2}P'$. Show that $K'K = A$, and explain why K is invertible — so every positive definite matrix can be written as $K'K$ for an invertible K .

SOLUTIONS

Since $D^{1/2}$ is diagonal, it is symmetric and $D^{1/2}D^{1/2} = D$. Then

$$K'K = (D^{1/2}P')'(D^{1/2}P') = P(D^{1/2})'D^{1/2}P' = PDP' = A.$$

$K = D^{1/2}P'$ is a product of invertible matrices: $D^{1/2}$ is invertible because each $\sqrt{\lambda_i} > 0$ (inverse $\text{diag}(1/\sqrt{\lambda_i})$), and P' is invertible because P is orthogonal ($(P')^{-1} = P$). Hence K is invertible, and every PD matrix factors as $A = K'K$. This is the matrix analog of writing a positive number as a square; the lower-triangular choice of such a factor is the Cholesky factor.

Question 14

Let A be symmetric with spectral decomposition $A = PDP'$, P orthogonal, $D = \text{diag}(\lambda_1, \dots, \lambda_n)$. Define new coordinates $z = P'x$.

- Show that the quadratic form satisfies $x'Ax = \sum_{i=1}^n \lambda_i z_i^2$.
- Use part (a) to prove that A is positive definite if and only if $\lambda_i > 0$ for all i .

SOLUTIONS

- Substitute $A = PDP'$ and $z = P'x$ (so $z' = x'P$):

$$x'Ax = x'PDP'x = (P'x)'D(P'x) = z'Dz = \sum_{i=1}^n \lambda_i z_i^2,$$

because D is diagonal.

- The map $x \mapsto z = P'x$ is a bijection (P invertible), with $z = 0 \iff x = 0$; also $\|z\|^2 = x'PP'x = x'x$, so $x \neq 0 \iff z \neq 0$.
 - ($\Leftarrow$) If every $\lambda_i > 0$: for $x \neq 0$ we have $z \neq 0$, so some $z_i \neq 0$ and $x'Ax = \sum_i \lambda_i z_i^2 > 0$. Thus A is PD.

- ($\Rightarrow$) If some $\lambda_j \leq 0$: take $x = Pe_j$ (the j -th eigenvector), giving $z = e_j$, so $x \neq 0$ but $x'Ax = \lambda_j \leq 0$. Then A is not PD. Contrapositively, PD forces all $\lambda_i > 0$.

PhD Track

Question 15

For any $n \times k$ matrix X , use the identity $c'X'Xc = \|Xc\|^2$ to prove that $X'X$ is positive definite if and only if X has full column rank.

- ($\Rightarrow$) Assuming X has full column rank, show $c'X'Xc > 0$ for all $c \neq 0$.
- ($\Leftarrow$) Assuming X is rank deficient, exhibit a specific $c \neq 0$ with $c'X'Xc = 0$.
- Combining with the eigenvalue characterization of definiteness, explain why $X'X$ is invertible if and only if X has full column rank.

SOLUTIONS

The identity $c'X'Xc = (Xc)'(Xc) = \|Xc\|^2 \geq 0$ shows $X'X$ is always positive semidefinite; strictness is the question.

- Full column rank means the columns are linearly independent, i.e. $Xc = 0 \Rightarrow c = 0$. So for $c \neq 0$, $Xc \neq 0$ and $c'X'Xc = \|Xc\|^2 > 0$. Hence $X'X$ is PD.
- Rank deficiency means the columns are dependent: there is a $c \neq 0$ with $Xc = 0$. For it, $c'X'Xc = \|Xc\|^2 = 0$ with $c \neq 0$, so $X'X$ is not PD (only PSD).
- A symmetric matrix is PD iff all eigenvalues are positive, and $\det(X'X) = \prod_i \lambda_i$, so $\text{PD} \Rightarrow \det(X'X) \neq 0 \Rightarrow X'X$ invertible. Conversely, if X is rank deficient, the $c \neq 0$ from (b) satisfies $X'Xc = X'(Xc) = 0$, so $X'X$ is singular. Therefore $X'X$ invertible $\Leftrightarrow X'X$ PD $\Leftrightarrow X$ has full column rank.