

Optimization — Practice Problems (Solutions)

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These practice questions accompany the Optimization lecture and will be discussed in the review session.

Question 1

Is the function $f(x) = x^2$ continuous? Why?

SOLUTIONS

Yes. Notice that $f(x + \varepsilon) = (x + \varepsilon)^2 = x^2 + 2x\varepsilon + \varepsilon^2$ which is very close to $f(x)$ when ε is small. (All polynomial functions are continuous.)

Question 2

Is the function

$$f(x) = \begin{cases} 0 & , \text{ if } x < 10 \\ 1 & , \text{ if } x \geq 10 \end{cases}$$

continuous?

SOLUTIONS

No. Notice that $f(10 + \varepsilon) = 1$ for any $\varepsilon > 0$ no matter how small ε is. So when ε becomes zero the function “jumps” from 0 to 1.

Question 3

Find the gradient of the following functions:

- a. $f(x) = 3(x + 10)^3 + 20x$.
- b. $f(x) = \log(\frac{x^2}{1+x})$ (You can assume $x > -1$).
- c. $f(x_1, x_2, x_3) = (3x_1x_2) \log(\frac{x_3+3}{2x_2})$ (You can assume $x_2 > 0, x_3 > -3$.)

SOLUTIONS

- a. This is a single variable function, so the gradient is just the derivative:

$$f'(x) = 9(x + 10)^2 + 20.$$

- b. This is also a single variable function, so the gradient is just the derivative: $f'(x) = \frac{2+x}{x(1+x)}$

- c. This is a multi-variable function, so the gradient is a vector:

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} \frac{\partial f(x_1, x_2, x_3)}{\partial x_1} \\ \frac{\partial f(x_1, x_2, x_3)}{\partial x_2} \\ \frac{\partial f(x_1, x_2, x_3)}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 3x_2 \log\left(\frac{x_3+3}{2x_2}\right) \\ 3x_1 \left[\log\left(\frac{x_3+3}{2x_2}\right) - 1 \right] \\ \frac{3x_1x_2}{x_3+3} \end{bmatrix}$$

Question 4

Determine whether the following functions are concave (strictly or not), convex (strictly or not) or neither:

- a. $f(x) = 3x_1 - 10$
- b. $f(x) = e^{x^2}$
- c. $f(x) = \log(2x_1 + 3)$
- d. $f(x) = 2x^3 - 3x^2 + 10x$

SOLUTIONS

- a. This is a linear function, so it is both concave and convex, but not strictly so (its second derivative is zero).
- b. This function is strictly convex since $f''(x) = 2e^{x^2}[1 + 2x^2] > 0$
- c. This function is strictly concave since $f''(x) = -\frac{4}{(2x_1+3)^2} < 0$.

- d. This function is neither concave nor convex since its second derivative is $f''(x) = 12x - 6$, which is strictly negative for $x < 0.5$ and strictly positive for $x > 0.5$.

Question 5

Function $f(x) = \log(1+x) - 3x$ has a global maximum. What is it? (You can assume throughout that $x > -1$.)

SOLUTIONS

Its global maximum must satisfy the first order condition $f'(x) = \frac{1}{1+x} - 3 = 0$, which gives $x^* = -\frac{2}{3}$.

Question 6

Function $f(x) = (2+x_1)x_2 - x_1 - x_2$ has a global maximum. What is it?

SOLUTIONS

Its global maximum must satisfy the first order condition $\nabla f(x) = [0, 0]^T$, which gives the system of equations:

$$\begin{aligned}\frac{\partial f(x)}{\partial x_1} &= x_2 - 1 = 0 \\ \frac{\partial f(x)}{\partial x_2} &= 2 + x_1 - 1 = 0\end{aligned}$$

This gives $x_1^* = -1$ and $x_2^* = 1$.

Question 7

Does the function $f(x) = -(x-2)^4 + 32x$ have a global maximum? If so, justify why and find this maximum.

SOLUTIONS

This function is strictly concave since its second derivative is $f''(x) = -12(x-2)^2 < 0$ for all points (except $x = 2$ where it is zero). So any point satisfying the first order condition $f'(x) = 0$ is a global maximum. The first order condition is $f'(x) = -4(x-2)^3 + 32 = 0$, which gives $x^* = 4$.

Question 8

Does the function $f(x) = x^4 + x$ have a global maximum? If so, justify why and find this maximum. Find the point satisfying the FOC. Why is it not a maximum?

SOLUTIONS

This function is strictly convex since its second derivative is $f''(x) = 12x^2 > 0$, so any point satisfying the first order condition is a global minimum instead of a global maximum. So this function does not have a global maximum.

Question 9

Find the global maximum of

$$f(x) = -3x^2 + 12x - 5.$$

Verify that the FOC is sufficient here.

SOLUTIONS

The FOC is $f'(x) = -6x + 12 = 0$, which gives $x^* = 2$.
Since $f''(x) = -6 < 0$ for every x , f is strictly concave everywhere, so the FOC is sufficient for a global maximum (no need to check any other candidate points).
The maximum value is $f(2) = -3(4) + 12(2) - 5 = -12 + 24 - 5 = 7$.

Question 10

Find all critical points of

$$f(x) = -x^4 + 8x^2.$$

Classify each one using the second order condition, and determine the global maximum.

SOLUTIONS

The FOC is

$$f'(x) = -4x^3 + 16x = -4x(x^2 - 4) = -4x(x - 2)(x + 2) = 0,$$

giving three critical points: $x = 0, 2, -2$.

The second derivative is $f''(x) = -12x^2 + 16$.

- At $x = 0$: $f''(0) = 16 > 0$, so this is a **local minimum** ($f(0) = 0$).
- At $x = 2$: $f''(2) = -48 + 16 = -32 < 0$, so this is a **local maximum** ($f(2) = -16 + 32 = 16$).
- At $x = -2$: $f''(-2) = -32 < 0$, so this is also a **local maximum** ($f(-2) = -16 + 32 = 16$, by symmetry since f is even).

Since $f(x) \rightarrow -\infty$ as $x \rightarrow \pm\infty$ (the leading term is $-x^4$), the function cannot exceed its largest local maximum anywhere else. The **global maximum is 16, attained at both $x^* = 2$ and $x^* = -2$** — note f is not strictly concave globally, so unlike the single-peaked function in Question 9, uniqueness fails and there are two global maximizers.

Question 11

Consider the same utility maximization problem as in the Optimization lecture. However, assume that utility function takes the form $a \ln(x) + b \ln(y)$. Solve this problem and discuss how its solution differs from that in the lecture.

SOLUTIONS

The Lagrangian is:

$$\Lambda(x, y, \lambda) = a \ln(x) + b \ln(y) + \lambda[m - p_x x - p_y y]$$

The first-order necessary conditions are:

$$\frac{\partial \Lambda}{\partial x} = \frac{a}{x} - \lambda p_x = 0$$

$$\frac{\partial \Lambda}{\partial y} = \frac{b}{y} - \lambda p_y = 0$$

$$\frac{\partial \Lambda}{\partial \lambda} = m - p_x x - p_y y = 0$$

Combining the 1st and 2nd conditions above gives $bxp_x = ayp_y$, which is the same condition as in the example in the lecture. Thus, the solution to this problem will be the same as the example solution. This is because log utility is just a log transformation of the Cobb-Douglas utility function.

Question 12

Minimize function $3x^2 + 2y^2 - z^2 + z$ subject to $x + y + z = 1$.

SOLUTIONS

The Lagrangian for this problem is

$$\Lambda(x, y, z, \lambda) = 3x^2 + 2y^2 - z^2 + z + \lambda(1 - x - y - z)$$

The first-order conditions are:

$$x : 6x - \lambda = 0$$

$$y : 4y - \lambda = 0$$

$$z : -2z + 1 - \lambda = 0$$

$$\lambda : 1 - x - y - z = 0$$

Solving this system of equations yields the following solution:

$$x = -1$$

$$y = -\frac{3}{2}$$

$$z = \frac{7}{2}$$

$$\lambda = -6$$

Question 13

Maximize function $xz + yz$ subject to constraints

$$y^2 + z^2 = 1$$

$$xz = 5$$

SOLUTIONS

The Lagrangian for this problem is

$$\Lambda(x, y, z, \lambda, \mu) = xz + yz + \lambda(1 - y^2 - z^2) + \mu(5 - xz)$$

The first-order conditions are

$$\begin{aligned}
x : z - \mu z &= 0 \\
y : z - 2y\lambda &= 0 \\
z : (y + x) - 2z\lambda - \mu x &= 0 \\
\lambda : 1 - y^2 - z^2 &= 0 \\
\mu : 5 - xz &= 0
\end{aligned}$$

Solving this system of equations gives the following solution:

$$\begin{aligned}
\mu &= 1 \\
\lambda &= \frac{z}{2y} \\
y = z &= \pm \sqrt{\frac{1}{2}} \\
x &= \frac{5}{z}.
\end{aligned}$$

Question 14

A household chooses quantities of three goods x, y, z to maximize Cobb-Douglas utility

$$u(x, y, z) = x^a y^b z^c$$

subject to **both** a budget constraint and a calorie constraint:

$$p_x x + p_y y + p_z z = m, \quad k_x x + k_y y + k_z z = K,$$

where p_x, p_y, p_z are prices, m is income, k_x, k_y, k_z are calories per unit of each good, and K is the household's required calorie intake.

Set up the Lagrangian and write down the first-order conditions.

SOLUTIONS

With two equality constraints, we attach one multiplier to each:

$$\Lambda(x, y, z, \lambda, \mu) = x^a y^b z^c + \lambda[m - p_x x - p_y y - p_z z] + \mu[K - k_x x - k_y y - k_z z].$$

The first-order conditions are

$$\begin{aligned}\frac{\partial \Lambda}{\partial x} &= a x^{a-1} y^b z^c - \lambda p_x - \mu k_x = 0, \\ \frac{\partial \Lambda}{\partial y} &= b x^a y^{b-1} z^c - \lambda p_y - \mu k_y = 0, \\ \frac{\partial \Lambda}{\partial z} &= c x^a y^b z^{c-1} - \lambda p_z - \mu k_z = 0,\end{aligned}$$

together with the two constraints themselves, $p_x x + p_y y + p_z z = m$ and $k_x x + k_y y + k_z z = K$. This is five equations in five unknowns (x, y, z, λ, μ) ; unlike the single-constraint Cobb-Douglas example, the system does not reduce to a clean closed-form ratio, so solving further generally requires plugging in numbers or using numerical methods. (Since both constraints are equalities, there is no guess-and-verify step here — both are automatically “binding” by construction.)

Question 15

Solve

$$\max_{x,y} -(x-6)^2 - (y-6)^2 \quad \text{s.t.} \quad x+y \leq 8, \quad x \leq 3.$$

Which constraint(s) bind at the optimum? Use the guess-and-verify procedure to justify your answer.

SOLUTIONS

The unconstrained optimum is $(x, y) = (6, 6)$, which violates **both** constraints ($6 + 6 = 12 > 8$ and $6 > 3$), so it’s a reasonable guess that both bind.

Guess: both constraints bind, i.e. $x = 3$ and $x + y = 8$, giving $y = 5$.

Writing the Lagrangian with λ_1 on $x + y \leq 8$ and λ_2 on $x \leq 3$:

$$\Lambda = -(x-6)^2 - (y-6)^2 + \lambda_1(8-x-y) + \lambda_2(3-x).$$

The FOCs are

$$-2(x-6) - \lambda_1 - \lambda_2 = 0, \quad -2(y-6) - \lambda_1 = 0.$$

At the candidate $(3, 5)$: the second FOC gives $-2(5-6) - \lambda_1 = 0 \Rightarrow \lambda_1 = 2$. The first FOC gives $-2(3-6) - 2 - \lambda_2 = 0 \Rightarrow 6 - 2 - \lambda_2 = 0 \Rightarrow \lambda_2 = 4$.

Both multipliers are positive ($\lambda_1^* = 2 > 0$, $\lambda_2^* = 4 > 0$), and the candidate satisfies both constraints with equality — so the guess is verified. **Both constraints bind** at $(x^*, y^*) = (3, 5)$, with objective value $-(3-6)^2 - (5-6)^2 = -9 - 1 = -10$.

PhD Track

Building on the guess-and-verify method for inequality constraints and on the PhD-track sections of the Optimization lecture (corner solutions and kinks, infinite-horizon optimization, and the envelope theorem).

PhD Track

Question 16

A decision-maker chooses a **state-contingent** consumption plan (c_1, c_2) : c_1 is consumed if state 1 occurs, c_2 if state 2 occurs, and the states occur with weights $\pi \in (0, 1)$ and $1 - \pi$ respectively. She solves

$$\max_{c_1, c_2} \pi u(c_1) + (1 - \pi) u(c_2) \quad \text{s.t.} \quad \pi c_1 + (1 - \pi) c_2 = W,$$

where u is differentiable, strictly increasing, and strictly concave, and $W > 0$.

- Set up the Lagrangian and derive the first-order conditions.
- Show that $u'(c_1^*) = u'(c_2^*)$, and use this to argue that $c_1^* = c_2^*$, **regardless of the value of π** .
- Solve for c_1^* and c_2^* in terms of W only.

SOLUTIONS

- The Lagrangian is

$$\Lambda(c_1, c_2, \lambda) = \pi u(c_1) + (1 - \pi) u(c_2) + \lambda[W - \pi c_1 - (1 - \pi) c_2].$$

The first-order conditions are

$$\begin{aligned} \frac{\partial \Lambda}{\partial c_1} &= \pi u'(c_1) - \lambda \pi = 0, \\ \frac{\partial \Lambda}{\partial c_2} &= (1 - \pi) u'(c_2) - \lambda(1 - \pi) = 0. \end{aligned}$$

- Since $\pi, 1 - \pi > 0$, both equations divide through to give $u'(c_1^*) = \lambda = u'(c_2^*)$. This holds for **any** $\pi \in (0, 1)$ — the weight cancels out of both FOCs because it multiplies both the marginal utility term and the shadow price term identically. Since u is strictly concave, u' is strictly decreasing, hence injective, so $u'(c_1^*) = u'(c_2^*)$ implies $c_1^* = c_2^*$.
- Let $c^* = c_1^* = c_2^*$. Substituting into the constraint: $\pi c^* + (1 - \pi) c^* = c^* = W$. So $c_1^* = c_2^* = W$, independent of π and of the specific functional form of u .

Question 17

Continue with the setup of Question 16, but now add a **subsistence floor** in state 2: consumption there cannot fall below some $\underline{c} > 0$, so the problem becomes

$$\max_{c_1, c_2} \pi u(c_1) + (1 - \pi) u(c_2) \quad \text{s.t.} \quad \pi c_1 + (1 - \pi) c_2 = W, \quad c_2 \geq \underline{c}.$$

- Suppose $\underline{c} < W$. Argue, without redoing the algebra, that the floor does not bind and the solution from Question 16 still applies.
- Now suppose $\underline{c} > W$ (and $W > (1 - \pi)\underline{c}$, so the problem remains feasible with $c_1 > 0$). Write the Lagrangian with a multiplier $\mu \geq 0$ on the floor constraint, solve for c_1^*, c_2^* , and find μ^* . Verify $\mu^* > 0$.

SOLUTIONS

- From Question 16, the unconstrained-floor solution is $c_1^* = c_2^* = W$. If $\underline{c} < W$, this candidate already satisfies $c_2^* = W \geq \underline{c}$, so the floor is slack at the candidate. A slack constraint can be dropped without changing the solution, so the answer from Question 16 still applies (with multiplier $\mu^* = 0$ on the floor).
- The Lagrangian is

$$\Lambda(c_1, c_2, \lambda, \mu) = \pi u(c_1) + (1 - \pi) u(c_2) + \lambda[W - \pi c_1 - (1 - \pi)c_2] + \mu[c_2 - \underline{c}].$$

If $\underline{c} > W$, the candidate from part (a) would require $c_2^* = W < \underline{c}$, violating feasibility — so guess instead that the floor binds: $c_2^* = \underline{c}$. The resource constraint then pins down

$$c_1^* = \frac{W - (1 - \pi)\underline{c}}{\pi},$$

which is positive by the assumption $W > (1 - \pi)\underline{c}$.

The FOCs are $\pi u'(c_1) - \lambda\pi = 0$ and $(1 - \pi)u'(c_2) - \lambda(1 - \pi) + \mu = 0$, giving $\lambda^* = u'(c_1^*)$ and

$$\mu^* = (1 - \pi)[u'(c_1^*) - u'(\underline{c})].$$

Since $\underline{c} > W$, one can check $c_1^* < \underline{c}$: indeed $c_1^* < \underline{c} \iff W - (1 - \pi)\underline{c} < \pi\underline{c} \iff W < \underline{c}$, which holds by assumption. Because u' is strictly decreasing, $c_1^* < \underline{c}$ implies $u'(c_1^*) > u'(\underline{c})$, so $\mu^* > 0$ as required — confirming the guess that the floor binds.

Question 18

Solve

$$\max_{x,y} -(x-4)^2 - (y-4)^2 \quad \text{s.t.} \quad x+y \leq 5, \quad x \geq 0, \quad y \geq 0.$$

- Guess that only the constraint $x+y \leq 5$ binds. Solve the resulting equality-constrained problem and verify the guess (feasibility of the dropped constraints and the sign of the multiplier).
- Now check the alternative guess that $x \geq 0$ binds instead (with $x+y \leq 5$ slack). Show that this guess fails verification, and explain which part of the KKT conditions it violates.

SOLUTIONS

- Writing the constraint as $x+y=5$, the Lagrangian is $\Lambda = -(x-4)^2 - (y-4)^2 + \lambda(5-x-y)$. The FOCs are

$$-2(x-4) - \lambda = 0, \quad -2(y-4) - \lambda = 0,$$

so $x = y = 4 - \lambda/2$. Substituting into $x+y=5$ gives $8-\lambda=5$, so $\lambda^* = 3$ and $x^* = y^* = 2.5$. Feasibility: $x^* = 2.5 \geq 0$ and $y^* = 2.5 \geq 0$, so both dropped constraints are indeed slack. Multiplier sign: $\lambda^* = 3 > 0$, as required. The guess is verified: $(x^*, y^*) = (2.5, 2.5)$.

- Write the constraint $x \geq 0$ as $-x \leq 0$ with multiplier $\mu \geq 0$, and drop $x+y \leq 5$ (guessed slack). The Lagrangian is $\Lambda = -(x-4)^2 - (y-4)^2 + \mu x$. The FOC for y (unconstrained) gives $y^* = 4$. The FOC for x at the candidate $x^* = 0$ gives

$$-2(x-4) + \mu = 0 \implies \mu = 2(x-4) \Big|_{x=0} = -8.$$

This is negative, violating **dual feasibility** ($\mu \geq 0$). A negative multiplier means that relaxing the constraint $x \geq 0$ (i.e. allowing $x < 0$) would *increase* the objective, so the agent would never want to sit at $x=0$ voluntarily — the guess is rejected. (One can also check $x+y = 0+4 = 4 \leq 5$, so feasibility alone does not rule out this guess; it is the multiplier sign that fails.)

PhD Track

Question 19

A consumer maximizes

$$u(x, y) = x + a \ln(y)$$

subject to $p_x x + p_y y = m$, with $x, y \geq 0$.

Set up the Lagrangian and derive the FOCs. For what values of m does the naive interior solution fail to satisfy $x \geq 0$? What is the optimal bundle in that case?

SOLUTIONS

The Lagrangian is $\Lambda = x + a \ln(y) + \lambda(m - p_x x - p_y y)$. The FOCs are

$$1 - \lambda p_x = 0 \Rightarrow \lambda = \frac{1}{p_x}, \quad \frac{a}{y} - \lambda p_y = 0 \Rightarrow y^* = \frac{ap_x}{p_y}.$$

Notice y^* does not depend on m — a hallmark of quasi-linear utility (all extra income is absorbed by x). Substituting into the budget constraint gives

$$x^* = \frac{m - p_y y^*}{p_x} = \frac{m}{p_x} - a.$$

This naive interior solution has $x^* \geq 0$ **iff** $m \geq ap_x$.

If $m < ap_x$: the interior solution is infeasible, and the true optimum is the corner $x^* = 0$, $y^* = m/p_y$ (spend everything on y). To confirm via KKT, write $x \geq 0$ as $-x \leq 0$ with multiplier $\mu \geq 0$: $\Lambda = x + a \ln y + \lambda(m - p_x x - p_y y) + \mu x$. At $x = 0, y = m/p_y$: the y -FOC gives $\lambda = \frac{a}{y^* p_y} = \frac{a}{m}$, and the x -FOC gives $\mu = \lambda p_x - 1 = \frac{ap_x - m}{m}$, which is **positive exactly when** $m < ap_x$ — confirming the corner is optimal in that range.

PhD Track

Question 20

A consumer maximizes

$$u(x, y) = \min\{ax, by\}$$

subject to $p_x x + p_y y = m$, with $x, y \geq 0$.

Why does the Lagrangian/FOC approach not apply directly here? Find the utility-maximizing bundle using economic reasoning instead.

SOLUTIONS

$u(x, y) = \min\{ax, by\}$ is not differentiable along the kink $ax = by$: on the region $ax < by$ the function equals ax (gradient $(a, 0)$), and on $ax > by$ it equals by (gradient $(0, b)$) — neither matches the tangency condition $\nabla u = \lambda \nabla g$ for any λ , and at the kink itself ∇u doesn't exist. So the smooth Lagrangian machinery cannot be applied directly.

Economic reasoning instead: away from the kink, one of the two “shares” ax or by is strictly larger and is being wasted (increasing the smaller share raises utility for free, since it doesn't reduce the larger one until they meet). So the optimum must sit exactly on the kink: $ax^* = by^*$, i.e. $y^* = \frac{a}{b}x^*$.

Substituting into the budget constraint:

$$p_x x^* + p_y \left(\frac{a}{b}x^*\right) = m \implies x^* \left(p_x + \frac{a}{b}p_y\right) = m \implies x^* = \frac{bm}{bp_x + ap_y}, \quad y^* = \frac{am}{bp_x + ap_y}.$$

PhD Track

Question 21

A consumer maximizes

$$u(x, y) = ax + by$$

subject to $p_x x + p_y y = m$, with $x, y \geq 0$.

Try applying the FOC $\nabla u(x^*) = \lambda \nabla g(x^*)$. Under what condition on prices does a well-defined interior solution exist? What happens otherwise?

SOLUTIONS

Here $\nabla u = (a, b)$ and $\nabla g = (p_x, p_y)$, both constant vectors (independent of x, y). The tangency condition $\nabla u = \lambda \nabla g$ requires

$$\frac{a}{p_x} = \frac{b}{p_y} \quad (= \lambda).$$

This is a single knife-edge condition on prices — generically it fails, and even when it holds exactly, the FOC is satisfied at **every** point on the budget line, not a unique one, so there is no well-defined interior solution either way.

Otherwise (the generic case): whichever good gives more utility per dollar wins the entire budget:

- If $a/p_x > b/p_y$: spend everything on x , i.e. $x^* = m/p_x$, $y^* = 0$.
- If $a/p_x < b/p_y$: spend everything on y , i.e. $y^* = m/p_y$, $x^* = 0$.

So the solution is always a corner, except on the knife-edge price ratio, where every bundle on the budget line is equally optimal.

PhD Track

Question 22

Consider the problem

$$\max_{x,y \geq 0} \max\{x, y\} \quad \text{s.t.} \quad x + y \leq w,$$

for some $w > 0$. This objective is a “worst-case” analogue of the perfect-substitutes case from Question 21: instead of adding x and y , we take whichever is larger.

- Show algebraically that $\max\{x, y\} \leq w$ for every feasible (x, y) , with equality if and only if $x + y = w$ and $\min\{x, y\} = 0$. Conclude that the maximizers are $(x, y) = (w, 0)$ and $(x, y) = (0, w)$.
- Try to apply the Lagrangian/FOC approach on the region where $x > y$ (so $u(x, y) = x$ there) and separately on the region where $y > x$. Show that the tangency condition $\nabla u = \lambda \nabla g$ has **no solution** in either region — unlike the perfect-substitutes case, where a matching condition on prices could be satisfied.
- Why does the FOC fail so completely here? Relate your answer to the concavity/convexity of u .

SOLUTIONS

- Since $x, y \geq 0$, $\max\{x, y\} \leq x + y$ always (the larger of two nonnegative numbers is at most their sum, with equality iff the smaller one is 0). Combined with the constraint $x + y \leq w$, this gives $\max\{x, y\} \leq x + y \leq w$. Equality throughout requires both $x + y = w$ (constraint binds) and $\min\{x, y\} = 0$ (one good is unconsumed). The only points satisfying both are $(w, 0)$ and $(0, w)$, at which $\max\{x, y\} = w$.
- On the region $x > y$, $u(x, y) = x$, so $\nabla u(x, y) = (1, 0)$. Writing the constraint as $g(x, y) = x + y$, $\nabla g = (1, 1)$. The tangency condition $\nabla u = \lambda \nabla g$ requires $(1, 0) = \lambda(1, 1)$, i.e. $\lambda = 1$ from the first component and $\lambda = 0$ from the second — a contradiction. By symmetry, on the region $y > x$ where $u(x, y) = y$, the same contradiction arises. And at the kink $x = y$, u is not differentiable, so ∇u does not exist at all. So **no point in the entire feasible set** — interior, boundary, or kink — satisfies the FOC. This is a stronger failure than perfect substitutes $u = ax + by$, where the FOC $\nabla u = \lambda \nabla g$ is at least *solvable*: it is just that it only holds along a knife-edge of prices. Here, no λ satisfies it anywhere.
- $u(x, y) = \max\{x, y\}$ is **convex**, not concave (it is the pointwise maximum of two

linear — hence convex — functions of (x, y) , and the max of convex functions is convex). Maximizing a convex function over a convex feasible set never occurs at an interior tangency point: a convex function attains its maximum over a compact convex set at an **extreme point** (corner) of that set, exactly as found in part (a). The Lagrangian/FOC machinery is built to find interior tangencies for concave objectives — applying it to a convex objective is the wrong tool entirely, not just a technical edge case like the kinks in the quasi-linear or perfect-complements examples.

PhD Track

Question 23

Suppose $u(c) = \ln(c)$ and $f(x) = x^\alpha$ for $\alpha \in (0, 1)$, with transition $x_{t+1} = x_t^\alpha - c_t$. Write out the Euler equation for this specific problem. What does it say about the relationship between c_t , c_{t+1} , and x_{t+1} ?

SOLUTIONS

The general Euler equation from the lecture is $u'(c_t) = \beta f'(x_{t+1}) u'(c_{t+1})$. Here $u'(c) = 1/c$ and $f'(x) = \alpha x^{\alpha-1}$, so

$$\frac{1}{c_t} = \beta \alpha x_{t+1}^{\alpha-1} \frac{1}{c_{t+1}} \iff c_{t+1} = \beta \alpha x_{t+1}^{\alpha-1} c_t.$$

Since $\alpha \in (0, 1)$, the exponent $\alpha - 1$ is negative, so $x_{t+1}^{\alpha-1}$ is a **decreasing** function of x_{t+1} : the more capital carried into next period, the lower its marginal product $f'(x_{t+1})$ (diminishing returns), and the smaller the required jump from c_t to c_{t+1} . A large capital stock next period therefore tempers consumption growth, consistent with the usual intuition that returns to saving fall as the capital stock rises.

PhD Track

Question 24

A household lives forever with discount factor $\beta \in (0, 1)$. Each period she receives fixed income $y > 0$ and holds assets a_t that earn gross return $R > 1$, so

$$a_{t+1} = Ra_t + y - c_t, \quad a_0 \text{ given,}$$

subject to a **no-borrowing condition** $a_{t+1} \geq 0$ for all $t \geq 0$. She maximizes $\sum_{t=0}^{\infty} \beta^t u(c_t)$ for some differentiable, strictly increasing, strictly concave u .

- a. Write the Lagrangian, attaching multiplier λ_t to the period- t budget constraint and

- $\mu_t \geq 0$ to the no-borrowing constraint $a_{t+1} \geq 0$.
- Derive the first-order conditions with respect to c_t and a_{t+1} .
 - Combine them into a single condition relating $u'(c_t)$ and $u'(c_{t+1})$. Show it holds as an equality (the usual Euler equation) when $a_{t+1} > 0$, and as the strict inequality $u'(c_t) \geq \beta R u'(c_{t+1})$ when the no-borrowing constraint binds.

SOLUTIONS

- The Lagrangian is

$$\Lambda = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t [Ra_t + y - c_t - a_{t+1}] + \sum_{t=0}^{\infty} \mu_t a_{t+1}.$$

- FOC w.r.t. c_t :** $\beta^t u'(c_t) - \lambda_t = 0 \implies \lambda_t = \beta^t u'(c_t)$.

FOC w.r.t. a_{t+1} : a_{t+1} appears as $-a_{t+1}$ in period t 's constraint, as Ra_{t+1} in period $t+1$'s constraint, and in the $\mu_t a_{t+1}$ term, so

$$-\lambda_t + \lambda_{t+1}R + \mu_t = 0 \implies \lambda_t = \lambda_{t+1}R + \mu_t.$$

- Substituting the first FOC into the second:

$$\beta^t u'(c_t) = \beta^{t+1} R u'(c_{t+1}) + \mu_t.$$

Dividing by $\beta^t > 0$:

$$u'(c_t) = \beta R u'(c_{t+1}) + \frac{\mu_t}{\beta^t}.$$

Since $\mu_t \geq 0$, this gives $u'(c_t) \geq \beta R u'(c_{t+1})$ always. By complementary slackness, $\mu_t a_{t+1} = 0$: if $a_{t+1} > 0$ then $\mu_t = 0$ and the condition holds with **equality** (the standard Euler equation); if the no-borrowing constraint binds ($a_{t+1} = 0$), then μ_t may be strictly positive and the inequality can be strict.

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Question 25

Suppose $u(c) = \ln(c)$, $a_0 = 0$, and $\beta R < 1$ (the household discounts the future faster than assets grow). Guess that the no-borrowing constraint binds in **every** period, i.e. $a_t = 0$ for all $t \geq 0$. Verify this guess using the condition from Question 24, and find the resulting consumption sequence.

SOLUTIONS

If $a_t = 0$ for all t , the budget constraint gives $c_t = Ra_t + y - a_{t+1} = y$ for every t : the household consumes exactly its income each period (“hand-to-mouth”).

To verify, check the condition from Question 24c at $c_t = c_{t+1} = y$:

$$u'(c_t) \geq \beta R u'(c_{t+1}) \iff \frac{1}{y} \geq \beta R \cdot \frac{1}{y} \iff 1 \geq \beta R,$$

which holds by assumption. The associated multiplier is $\mu_t = \beta^t [u'(c_t) - \beta R u'(c_{t+1})] = \beta^t \frac{1 - \beta R}{y} > 0$, confirming that the constraint is indeed strictly binding in every period, consistent with the guess. Intuitively: the household is impatient enough ($\beta R < 1$) that it would like to borrow against future income to consume more today, but the no-borrowing condition prevents this, pinning consumption to income in every period.

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Question 26

Let $f(x, \theta) = -(x - \theta)^2 + \theta x$ and define $V(\theta) = \max_x f(x, \theta)$.

- Solve for $x^*(\theta)$, compute $V(\theta)$ explicitly, and find $dV/d\theta$ by direct differentiation.
- Compute $\partial f / \partial \theta$ holding x fixed, then evaluate it at $x = x^*(\theta)$. Confirm this matches part (a), as predicted by the envelope theorem.

SOLUTIONS

- The FOC is $\frac{\partial f}{\partial x} = -2(x - \theta) + \theta = 0 \implies -2x + 3\theta = 0 \implies x^*(\theta) = \frac{3\theta}{2}$.

Then

$$V(\theta) = -\left(\frac{3\theta}{2} - \theta\right)^2 + \theta \cdot \frac{3\theta}{2} = -\frac{\theta^2}{4} + \frac{3\theta^2}{2} = \frac{5\theta^2}{4},$$

so $\frac{dV}{d\theta} = \frac{5\theta}{2}$.

- Holding x fixed, $\frac{\partial f}{\partial \theta} = 2(x - \theta) + x$. Evaluating at $x = x^*(\theta) = \frac{3\theta}{2}$:

$$\left. \frac{\partial f}{\partial \theta} \right|_{x^*} = 2\left(\frac{3\theta}{2} - \theta\right) + \frac{3\theta}{2} = 2 \cdot \frac{\theta}{2} + \frac{3\theta}{2} = \theta + \frac{3\theta}{2} = \frac{5\theta}{2}.$$

This matches $dV/d\theta$ from part (a), exactly as the envelope theorem predicts.

Question 27

Let $V(\theta) = \max_x \ln(x)$ s.t. $\theta x = 1$, for $\theta > 0$. Note that here the parameter θ appears **inside the constraint function**, not just on the right-hand side.

- Solve for $x^*(\theta)$ and $V(\theta)$ explicitly, and find $dV/d\theta$ by direct differentiation.
- Apply the envelope theorem to the Lagrangian $\Lambda(x, \lambda; \theta) = \ln(x) + \lambda(1 - \theta x)$ to compute $dV/d\theta$. Confirm the two answers agree.

SOLUTIONS

- The constraint gives $x^*(\theta) = 1/\theta$, so $V(\theta) = \ln(1/\theta) = -\ln(\theta)$, and

$$\frac{dV}{d\theta} = -\frac{1}{\theta}.$$

- The FOC is $\frac{1}{x} - \lambda\theta = 0 \implies \lambda = \frac{1}{x\theta}$. At $x^*(\theta) = 1/\theta$, this gives $\lambda^*(\theta) = \frac{1}{(1/\theta)\theta} = 1$.

Holding x and λ fixed and differentiating Λ with respect to θ (product rule on $\lambda(1 - \theta x)$, noting θ now multiplies x inside the constraint):

$$\frac{\partial \Lambda}{\partial \theta} = \lambda \cdot (-x).$$

Evaluating at $(x^*, \lambda^*) = (1/\theta, 1)$:

$$\left. \frac{\partial \Lambda}{\partial \theta} \right|_{(x^*, \lambda^*)} = 1 \cdot \left(-\frac{1}{\theta} \right) = -\frac{1}{\theta}.$$

This matches $dV/d\theta$ from part (a). The envelope theorem holds regardless of *where* θ enters the problem — objective or constraint — as long as we differentiate the full Lagrangian and hold the optimizing variables fixed.