

Probability — Practice Problems

Vadim Marmer

Paul Schrimpf

These practice questions accompany the Probability lecture and will be discussed in the review session.

Question 1

For the rolling of a die experiment, define $X(\omega) = \omega$ for $\omega \in \Omega = \{1, 2, 3, 4, 5, 6\}$. Construct the CDF $F(x)$ of X .

Question 2

Suppose X is uniformly distributed on $[a, b]$: the PDF is a strictly positive constant on $[a, b]$, and is zero otherwise. Construct the CDF and PDF of X .

Question 3

The joint PDF of X and Y is $f(x, y) = ax \cdot \mathbb{1}(0 < x < 1, x < y < 1)$, where $\mathbb{1}(\cdot)$ is the indicator function that takes the value 1 if the condition is true and 0 otherwise.

- Find a such that $f(x, y)$ is a valid PDF
- Find the marginal PDF of X
- Find the marginal PDF of Y
- Are X and Y independent?

Question 4

Suppose X is a Bernoulli random variable: $\mathbb{P}[X = 1] = p$, $\mathbb{P}[X = 0] = 1 - p$ for $p \in (0, 1)$.

- Compute the first and second moments of X
- Compute the variance of X

Question 5

Suppose $\mathbb{E}[X] = \mathbb{E}[Y]$ for random variables X and Y . Does this imply that $X = Y$? Prove if true or give a counterexample if false.

Question 6

For the joint PDF from Question 3, compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathbb{E}[XY]$, and $\text{Cov}(X, Y)$.

Question 7

Compute the quantiles of order τ for the $\text{Uniform}(a, b)$ distribution.

Question 8

Suppose $Y = g(X)$, where g is a strictly increasing function. Show that $q_\tau^Y = g(q_\tau^X)$, where q_τ^X and q_τ^Y are the quantiles of X and Y , respectively.

Question 9

Using the Markov inequality, show that $\mathbb{P}[|X - \mathbb{E}[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2}$ for any $a > 0$.

Question 10

The correlation coefficient ρ_{XY} between the random variables X and Y is defined as $\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}$. Show that $|\rho_{XY}| \leq 1$ using the Cauchy-Schwarz inequality.

Question 11

Using Jensen's inequality, show that if $\mathbb{E}[|X|^p] < \infty$ for some $p > 0$, then $\mathbb{E}[|X|^r] < \infty$ for all $r \in [0, p]$. Hint: $|x|^s$ is concave for $0 < s < 1$.

Question 12

A disease affects 1% of the population: $\mathbb{P}[D] = 0.01$. A diagnostic test has sensitivity $\mathbb{P}[T^+ | D] = 0.95$ and specificity $\mathbb{P}[T^- | D^c] = 0.90$. A randomly chosen person tests positive. Using the multiplication rule, the law of total probability, and Bayes' rule, compute the posterior $\mathbb{P}[D | T^+]$.

Question 13

The joint PMF of discrete $X_1 \in \{1, 2\}$ and $X_2 \in \{1, 2, 3\}$ is

$p(x_1, x_2)$	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$
$X_1 = 1$	0.10	0.15	0.05
$X_1 = 2$	0.20	0.30	0.20

Compute the marginal PMFs $p_{X_1}(x_1)$ and $p_{X_2}(x_2)$, and verify each sums to 1.

Question 14

Let $X \in \{0, 1\}$ and $Y \in \{0, 1, 2\}$ have joint PMF

$p(x, y)$	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	0.10	0.20	0.10
$X = 1$	0.05	0.25	0.30

Compute $\mathbb{E}[Y \mid X = 0]$ and $\mathbb{E}[Y \mid X = 1]$, and express $\mathbb{E}[Y \mid X]$ as a function of X .

Question 15

Prove the “taking out what is known” property for discrete random variables: for any function g ,

$$\mathbb{E}[g(X)Y \mid X] = g(X)\mathbb{E}[Y \mid X].$$

Question 16

A course has two sections: A has 30 students averaging 72, and B has 20 students averaging 80. With S the section and G the grade, $\mathbb{E}[G \mid S = A] = 72$ and $\mathbb{E}[G \mid S = B] = 80$. Use the law of iterated expectations to find $\mathbb{E}[G]$.

Question 17

Using the joint PMF of $X \in \{0, 1\}$, $Y \in \{0, 1, 2\}$ from Question 14,

$p(x, y)$	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	0.10	0.20	0.10
$X = 1$	0.05	0.25	0.30

compute $\mathbb{E}[Y]$ two ways and confirm they agree: (a) directly from the marginal p_Y ; (b) via the law of iterated expectations.

Question 18

Prove the Cauchy–Schwarz inequality: for random variables X, Y with finite second moments,

$$(\mathbb{E}[XY])^2 \leq \mathbb{E}[X^2] \mathbb{E}[Y^2].$$

Hint: consider $\mathbb{E}[(X - tY)^2] \geq 0$ as a quadratic in the real variable t .