

Probability — Practice Problems (Solutions)

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These practice questions accompany the Probability lecture and will be discussed in the review session.

Question 1

For the rolling of a die experiment, define $X(\omega) = \omega$ for $\omega \in \Omega = \{1, 2, 3, 4, 5, 6\}$. Construct the CDF $F(x)$ of X .

SOLUTIONS

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/6 & 1 \leq x < 2 \\ 2/6 & 2 \leq x < 3 \\ 3/6 & 3 \leq x < 4 \\ 4/6 & 4 \leq x < 5 \\ 5/6 & 5 \leq x < 6 \\ 1 & x \geq 6 \end{cases}$$

Question 2

Suppose X is uniformly distributed on $[a, b]$: the PDF is a strictly positive constant on $[a, b]$, and is zero otherwise. Construct the CDF and PDF of X .

SOLUTIONS

PDF and CDF of $X \sim \text{Uniform}(a, b)$:

$$f(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a \leq x \leq b \\ 0 & x > b \end{cases}$$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

Question 3

The joint PDF of X and Y is $f(x, y) = ax \cdot \mathbb{1}(0 < x < 1, x < y < 1)$, where $\mathbb{1}(\cdot)$ is the indicator function that takes the value 1 if the condition is true and 0 otherwise.

- Find a such that $f(x, y)$ is a valid PDF
- Find the marginal PDF of X
- Find the marginal PDF of Y
- Are X and Y independent?

SOLUTIONS

- A valid PDF must integrate to 1: $\int_{x=0}^1 \int_{y=x}^1 ax \, dy \, dx = 1$. The inner integral is $\int_{y=x}^1 dy = 1 - x$, so $\int_{x=0}^1 ax(1-x) \, dx = a \int_{x=0}^1 x(1-x) \, dx$. But $\int_{x=0}^1 x(1-x) \, dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, so $a \cdot \frac{1}{6} = 1 \implies a = 6$
- $f_X(x) = \int_{y=x}^1 f(x, y) \, dy = \int_{y=x}^1 6x \, dy = 6x(1-x)$ for $0 < x < 1$. The PDF is zero for $x \leq 0$ or $x \geq 1$
- $f_Y(y) = \int_{x=0}^y f(x, y) \, dx = \int_{x=0}^y 6x \, dx = 6 \cdot \frac{y^2}{2} = 3y^2$ for $0 < y < 1$. The PDF is zero for $y \leq 0$ or $y \geq 1$
- X and Y are not independent: the indicator function $\mathbb{1}(x < y < 1)$ cannot be written as a product of functions of x and y alone. Thus, the joint PDF cannot be factored into the product of the marginal PDFs

Question 4

Suppose X is a Bernoulli random variable: $\mathbb{P}[X = 1] = p$, $\mathbb{P}[X = 0] = 1 - p$ for $p \in (0, 1)$.

- Compute the first and second moments of X

b. Compute the variance of X

SOLUTIONS

a. $\mathbb{E}[X] = p, \mathbb{E}[X^2] = p$

b. $\text{Var}(X) = p - p^2 = p(1 - p)$

Question 5

Suppose $\mathbb{E}[X] = \mathbb{E}[Y]$ for random variables X and Y . Does this imply that $X = Y$? Prove if true or give a counterexample if false.

SOLUTIONS

The claim “ $\mathbb{E}[X] = \mathbb{E}[Y]$ implies $X = Y$ ” is **false**. Counterexample: Let $X \sim \text{Bernoulli}(1/2)$ and $Y = 1 - X$. Then $\mathbb{E}[X] = \mathbb{E}[Y] = 1/2$, but $\mathbb{P}[X = Y] = 0$, so $X \neq Y$ almost surely.

Question 6

For the joint PDF from Question 3, compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathbb{E}[XY]$, and $\text{Cov}(X, Y)$.

SOLUTIONS

For $f(x, y) = 6x \cdot \mathbb{1}(0 < x < 1, x < y < 1)$:

$$\mathbb{E}[X] = \int_0^1 \int_x^1 x \cdot 6x \, dy \, dx = \int_0^1 6x^2(1 - x) \, dx = 1/2$$

$$\mathbb{E}[Y] = \int_0^1 \int_0^y y \cdot 6x \, dx \, dy = \int_0^1 y \cdot 3y^2 \, dy = 3/4$$

$$\mathbb{E}[XY] = \int_0^1 \int_x^1 xy \cdot 6x \, dy \, dx = \int_0^1 3x^2(1 - x^2) \, dx = 2/5$$

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 2/5 - (1/2)(3/4) = 1/40$$

Question 7

Compute the quantiles of order τ for the $\text{Uniform}(a, b)$ distribution.

SOLUTIONS

For $X \sim \text{Uniform}(a, b)$, the CDF is $F_X(x) = 0$ for $x < a$, $F_X(x) = \frac{x-a}{b-a}$ for $a \leq x \leq b$, and $F_X(x) = 1$ for $x > b$. By definition, $q_\tau = a + \tau(b-a)$ for $0 < \tau < 1$; $q_0 = a$, $q_1 = b$

Question 8

Suppose $Y = g(X)$, where g is a strictly increasing function. Show that $q_\tau^Y = g(q_\tau^X)$, where q_τ^X and q_τ^Y are the quantiles of X and Y , respectively.

SOLUTIONS

$\mathbb{P}[Y \leq g(q_\tau^X)] = \mathbb{P}[g(X) \leq g(q_\tau^X)] = \mathbb{P}[X \leq q_\tau^X] = \tau$, so $q_\tau^Y = g(q_\tau^X)$, where the second equality holds because g is strictly increasing, preserving the order of inequalities

Question 9

Using the Markov inequality, show that $\mathbb{P}[|X - \mathbb{E}[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2}$ for any $a > 0$.

SOLUTIONS

Let $Z = (X - \mathbb{E}[X])^2 \geq 0$. By Markov, $\mathbb{P}[|X - \mathbb{E}[X]| \geq a] = \mathbb{P}[Z \geq a^2] \leq \mathbb{E}[Z]/a^2 = \text{Var}(X)/a^2$ for any $a > 0$

Question 10

The correlation coefficient ρ_{XY} between the random variables X and Y is defined as $\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}$. Show that $|\rho_{XY}| \leq 1$ using the Cauchy-Schwarz inequality.

SOLUTIONS

By Cauchy-Schwarz, $|\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]| \leq \sqrt{\mathbb{E}[(X - \mathbb{E}[X])^2] \mathbb{E}[(Y - \mathbb{E}[Y])^2]}$, hence $|\text{Cov}(X, Y)| \leq \sqrt{\text{Var}(X) \text{Var}(Y)}$ and $|\rho_{XY}| \leq 1$ (assuming positive, finite variances)

Question 11

Using Jensen's inequality, show that if $\mathbb{E}[|X|^p] < \infty$ for some $p > 0$, then $\mathbb{E}[|X|^r] < \infty$ for all $r \in [0, p]$. Hint: $|x|^s$ is concave for $0 < s < 1$.

SOLUTIONS

$|x|^r = (|x|^p)^{r/p}$, where $(\cdot)^{r/p}$ is concave for $0 < r < p$. Therefore, $\mathbb{E}[|X|^r] = \mathbb{E}[(|X|^p)^{r/p}] \leq (\mathbb{E}[|X|^p])^{r/p} < \infty$ for $0 < r < p$, where the first inequality follows by Jensen's and the second by the assumption $\mathbb{E}[|X|^p] < \infty$.

Question 12

A disease affects 1% of the population: $\mathbb{P}[D] = 0.01$. A diagnostic test has sensitivity $\mathbb{P}[T^+ | D] = 0.95$ and specificity $\mathbb{P}[T^- | D^c] = 0.90$. A randomly chosen person tests positive. Using the multiplication rule, the law of total probability, and Bayes' rule, compute the posterior $\mathbb{P}[D | T^+]$.

SOLUTIONS

From specificity, the false-positive rate is $\mathbb{P}[T^+ | D^c] = 1 - 0.90 = 0.10$, and $\mathbb{P}[D^c] = 0.99$.
Law of total probability:

$$\mathbb{P}[T^+] = \mathbb{P}[T^+ | D]\mathbb{P}[D] + \mathbb{P}[T^+ | D^c]\mathbb{P}[D^c] = (0.95)(0.01) + (0.10)(0.99) = 0.0095 + 0.099 = 0.1085.$$

Bayes' rule:

$$\mathbb{P}[D | T^+] = \frac{\mathbb{P}[T^+ | D]\mathbb{P}[D]}{\mathbb{P}[T^+]} = \frac{0.0095}{0.1085} = \frac{19}{217} \approx 0.088.$$

Even with 95% sensitivity, only about **8.8%** of positive testers have the disease: because the disease is rare, false positives from the healthy 99% far outnumber true positives. This base-rate effect is the main lesson.

Question 13

The joint PMF of discrete $X_1 \in \{1, 2\}$ and $X_2 \in \{1, 2, 3\}$ is

$p(x_1, x_2)$	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$
$X_1 = 1$	0.10	0.15	0.05
$X_1 = 2$	0.20	0.30	0.20

Compute the marginal PMFs $p_{X_1}(x_1)$ and $p_{X_2}(x_2)$, and verify each sums to 1.

SOLUTIONS

Marginal of X_1 (sum across each row):

$$p_{X_1}(1) = 0.10 + 0.15 + 0.05 = 0.30, \quad p_{X_1}(2) = 0.20 + 0.30 + 0.20 = 0.70,$$

and $0.30 + 0.70 = 1$. ✓

Marginal of X_2 (sum down each column):

$$p_{X_2}(1) = 0.30, \quad p_{X_2}(2) = 0.45, \quad p_{X_2}(3) = 0.25,$$

and $0.30 + 0.45 + 0.25 = 1$. ✓

Question 14

Let $X \in \{0, 1\}$ and $Y \in \{0, 1, 2\}$ have joint PMF

$p(x, y)$	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	0.10	0.20	0.10
$X = 1$	0.05	0.25	0.30

Compute $\mathbb{E}[Y \mid X = 0]$ and $\mathbb{E}[Y \mid X = 1]$, and express $\mathbb{E}[Y \mid X]$ as a function of X .

SOLUTIONS

Marginal of X : $p_X(0) = 0.40$, $p_X(1) = 0.60$.

Conditional on $X = 0$: $p_{Y|X}(\cdot \mid 0) = (0.25, 0.50, 0.25)$, so

$$\mathbb{E}[Y \mid X = 0] = 0(0.25) + 1(0.50) + 2(0.25) = 1.$$

Conditional on $X = 1$: $p_{Y|X}(\cdot \mid 1) = (\frac{1}{12}, \frac{5}{12}, \frac{6}{12})$, so

$$\mathbb{E}[Y \mid X = 1] = 1 \cdot \frac{5}{12} + 2 \cdot \frac{6}{12} = \frac{17}{12} \approx 1.417.$$

As a random variable, $\mathbb{E}[Y \mid X] = g(X)$ with $g(0) = 1$ and $g(1) = \frac{17}{12}$. Since X is binary, equivalently $\mathbb{E}[Y \mid X] = 1 + \frac{5}{12}X$.

Question 15

Prove the “taking out what is known” property for discrete random variables: for any function g ,

$$\mathbb{E}[g(X)Y \mid X] = g(X)\mathbb{E}[Y \mid X].$$

SOLUTIONS

Fix a value x with $p_X(x) > 0$. Conditional on $X = x$, the factor $g(X)$ is the constant $g(x)$, so

$$\mathbb{E}[g(X)Y \mid X = x] = \sum_y g(x) y p_{Y|X}(y \mid x) = g(x) \sum_y y p_{Y|X}(y \mid x) = g(x) \mathbb{E}[Y \mid X = x].$$

The constant $g(x)$ pulls out of the sum because it does not depend on y . Since this holds for every value x of X , the random variables agree: $\mathbb{E}[g(X)Y \mid X] = g(X) \mathbb{E}[Y \mid X]$. Once we condition on X , anything that is a function of X is known and factors out.

Question 16

A course has two sections: A has 30 students averaging 72, and B has 20 students averaging 80. With S the section and G the grade, $\mathbb{E}[G \mid S = A] = 72$ and $\mathbb{E}[G \mid S = B] = 80$. Use the law of iterated expectations to find $\mathbb{E}[G]$.

SOLUTIONS

With $30 + 20 = 50$ students, the section probabilities are $\mathbb{P}[S = A] = 0.6$ and $\mathbb{P}[S = B] = 0.4$. By the law of iterated expectations,

$$\mathbb{E}[G] = \mathbb{E}[G \mid S = A]\mathbb{P}[S = A] + \mathbb{E}[G \mid S = B]\mathbb{P}[S = B] = 72(0.6) + 80(0.4) = 43.2 + 32 = 75.2.$$

This is a weighted average of the section means, pulled toward 72 because section A is larger; the naive average $(72 + 80)/2 = 76$ would over-weight the smaller section.

Question 17

Using the joint PMF of $X \in \{0, 1\}$, $Y \in \{0, 1, 2\}$ from Question 14,

$p(x, y)$	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	0.10	0.20	0.10
$X = 1$	0.05	0.25	0.30

compute $\mathbb{E}[Y]$ two ways and confirm they agree: (a) directly from the marginal p_Y ; (b) via the law of iterated expectations.

SOLUTIONS

(a) **Directly.** Marginal of Y : $p_Y(0) = 0.15$, $p_Y(1) = 0.45$, $p_Y(2) = 0.40$. Then

$$\mathbb{E}[Y] = 0(0.15) + 1(0.45) + 2(0.40) = 1.25.$$

(b) **Via the LIE.** From Question 14, $\mathbb{E}[Y \mid X = 0] = 1$, $\mathbb{E}[Y \mid X = 1] = \frac{17}{12}$, with $p_X(0) = 0.40$, $p_X(1) = 0.60$:

$$\mathbb{E}[Y] = 1(0.40) + \frac{17}{12}(0.60) = 0.40 + 0.85 = 1.25.$$

Both give $\mathbb{E}[Y] = 1.25$, confirming $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]]$. ✓

Question 18

Prove the Cauchy–Schwarz inequality: for random variables X, Y with finite second moments,

$$(\mathbb{E}[XY])^2 \leq \mathbb{E}[X^2] \mathbb{E}[Y^2].$$

Hint: consider $\mathbb{E}[(X - tY)^2] \geq 0$ as a quadratic in the real variable t .

SOLUTIONS

For every real t , $(X - tY)^2 \geq 0$, so

$$q(t) := \mathbb{E}[(X - tY)^2] = \mathbb{E}[X^2] - 2t \mathbb{E}[XY] + t^2 \mathbb{E}[Y^2] \geq 0 \quad \text{for all } t.$$

If $\mathbb{E}[Y^2] = 0$ then $Y = 0$ almost surely and both sides vanish. Otherwise q is a quadratic in t with positive leading coefficient that never goes negative, so its discriminant is nonpositive:

$$(2 \mathbb{E}[XY])^2 - 4 \mathbb{E}[Y^2] \mathbb{E}[X^2] \leq 0 \implies (\mathbb{E}[XY])^2 \leq \mathbb{E}[X^2] \mathbb{E}[Y^2].$$

Equality holds iff $q(t) = 0$ for some t , i.e. $X = tY$ almost surely. Applying the inequality to the centered variables $X - \mathbb{E}[X]$ and $Y - \mathbb{E}[Y]$ gives $|\text{Cov}(X, Y)| \leq \sqrt{\text{Var}(X) \text{Var}(Y)}$, hence $|\rho_{XY}| \leq 1$ (compare Question 10).