

Proof and Real Analysis — Practice Problems

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These practice questions accompany the Proof and Real Analysis lecture. Each asks you to prove a result stated in the lecture. A few of the harder proofs are marked (*Challenging*) — attempt the others first.

Question 1

Prove De Morgan's laws for sets: for $A, B \subseteq X$,

$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c.$$

Question 2

Prove the following stability properties of open and closed sets in $\mathbb{R}^n$.

- The union of an arbitrary family of open sets is open.
- The intersection of finitely many open sets is open.
- Using De Morgan's laws, deduce that an arbitrary intersection of closed sets is closed, and a finite union of closed sets is closed.

Question 3

Let $A \subseteq \mathbb{R}$ be non-empty and let $s \in \mathbb{R}$ be an upper bound of A . Prove that $s = \sup A$ if and only if

$$\forall \varepsilon > 0, \exists a \in A \text{ such that } a > s - \varepsilon.$$

Question 4

Using the definition **compact** = **closed and bounded** (in $\mathbb{R}^n$), justify each claim.

- $[0, 1]^n$ is compact.
- $(0, 1)$ is not compact.
- $\mathbb{R}$ is not compact.
- The budget set $\{x \in \mathbb{R}^n : x \geq 0, p \cdot x \leq w\}$ is compact, where $p \gg 0$ (all prices strictly positive) and $w > 0$.

Question 5

Prove that if a sequence (x_k) in $\mathbb{R}^n$ converges, then its limit is unique.

Question 6

Let $x_k \rightarrow x$ and $y_k \rightarrow y$ in $\mathbb{R}$. Prove:

- (Algebraic Limit Theorem)** $x_k + y_k \rightarrow x + y$; $cx_k \rightarrow cx$; $x_k y_k \rightarrow xy$; and, if $y \neq 0$ and $y_k \neq 0$, $x_k/y_k \rightarrow x/y$.
- (Order Limit Theorem)** if $x_k \leq y_k$ for all k , then $x \leq y$.
- (Squeeze Theorem)** if $x_k \leq z_k \leq y_k$ for all k and $x_k, y_k \rightarrow L$, then $z_k \rightarrow L$.

The quotient limit in (a) is the most delicate part.

Question 7

- Prove the **Monotone Convergence Theorem**: every bounded, monotone sequence in $\mathbb{R}$ converges. (Prove the non-decreasing case; the non-increasing case is symmetric.)
- Show **directly** from the ε - K definition — without invoking the MCT — that $x_k = 1 - \frac{1}{k}$ converges to 1.

Question 8

- Prove the **sequential characterisation of closed sets**: $F \subseteq \mathbb{R}^n$ is closed if and only if every convergent sequence $(x_k) \subseteq F$ has its limit in F .
- Use it to show $[0, 1]$ is closed and $(0, 1)$ is not.

Question 9

- Prove that if $x_k \rightarrow L$, then every subsequence (x_{k_j}) converges to L .
- (Challenging.) Prove the **Bolzano–Weierstrass theorem**: every bounded sequence in $\mathbb{R}$ has a convergent subsequence. *Hint: first show every real sequence has a monotone subsequence, then apply the Monotone Convergence Theorem.*

Question 10

Let $f : X \rightarrow Y$ and $A, B \subseteq Y$. Prove the preimage identities:

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), \quad f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B), \quad f^{-1}(A^c) = (f^{-1}(A))^c.$$

Question 11

Prove that the ε - δ and sequential definitions of continuity at a point are equivalent: for $f : X \rightarrow \mathbb{R}^m$ and $x_0 \in X$,

$$(\forall \varepsilon > 0 \exists \delta > 0 : \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon) \iff (x_k \rightarrow x_0 \Rightarrow f(x_k) \rightarrow f(x_0)).$$

Question 12

Let $f, g : X \rightarrow \mathbb{R}$ be continuous at x_0 , and let $c \in \mathbb{R}$.

- Using the sequential characterisation of continuity, prove that $f + g$, cf , and fg are continuous at x_0 , and that f/g is continuous at x_0 provided $g(x_0) \neq 0$.
- Prove that a composition of continuous functions is continuous: if $f : X \rightarrow Y$ is continuous at x_0 and $g : Y \rightarrow Z$ is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0 .

Question 13

(Challenging.)

- Prove the **Extreme Value Theorem**: if $K \subseteq \mathbb{R}^n$ is compact and $f : K \rightarrow \mathbb{R}$ is continuous, then f attains a maximum and a minimum on K .
- Prove the **Intermediate Value Theorem**: if $f : [a, b] \rightarrow \mathbb{R}$ is continuous and y lies between $f(a)$ and $f(b)$, then $f(c) = y$ for some $c \in [a, b]$.