

Proof and Real Analysis — Practice Problems (Solutions)

William Duan

These practice questions accompany the Proof and Real Analysis lecture. Each asks you to prove a result stated in the lecture. A few of the harder proofs are marked (*Challenging*) — attempt the others first.

Question 1

Prove De Morgan's laws for sets: for $A, B \subseteq X$,

$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c.$$

SOLUTIONS

We prove each equality by a chain of “if and only if” statements on membership.

First law. For any $x \in X$,

$$x \in (A \cup B)^c \iff x \notin A \cup B \iff (x \notin A \text{ and } x \notin B) \iff x \in A^c \cap B^c.$$

The middle step is the negation of “ $x \in A$ or $x \in B$ ”, which is “ $x \notin A$ and $x \notin B$ ”. Since x was arbitrary, $(A \cup B)^c = A^c \cap B^c$.

Second law. Similarly,

$$x \in (A \cap B)^c \iff \neg(x \in A \text{ and } x \in B) \iff (x \notin A \text{ or } x \notin B) \iff x \in A^c \cup B^c.$$

Hence $(A \cap B)^c = A^c \cup B^c$. (Alternatively, this follows from the first law applied to A^c, B^c together with $(E^c)^c = E$.)

Question 2

Prove the following stability properties of open and closed sets in $\mathbb{R}^n$.

- The union of an arbitrary family of open sets is open.
- The intersection of finitely many open sets is open.
- Using De Morgan's laws, deduce that an arbitrary intersection of closed sets is closed, and a finite union of closed sets is closed.

SOLUTIONS

- Let $\{U_i\}_{i \in I}$ be open and $U = \bigcup_{i \in I} U_i$. Take $x \in U$; then $x \in U_{i_0}$ for some $i_0 \in I$. Since U_{i_0} is open, there is $\varepsilon > 0$ with $B(x, \varepsilon) \subseteq U_{i_0} \subseteq U$. As $x \in U$ was arbitrary, U is open. (No finiteness is needed here.)
- Let $U_1, \dots, U_n$ be open and $U = \bigcap_{k=1}^n U_k$. Take $x \in U$, so $x \in U_k$ for every k . For each k choose $\varepsilon_k > 0$ with $B(x, \varepsilon_k) \subseteq U_k$, and set $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$. The minimum of finitely many positive numbers is positive, so $\varepsilon > 0$, and $B(x, \varepsilon) \subseteq B(x, \varepsilon_k) \subseteq U_k$ for every k ; hence $B(x, \varepsilon) \subseteq U$. So U is open. (Finiteness is essential: it is what guarantees $\varepsilon > 0$.)
- Let $\{F_i\}_{i \in I}$ be closed, so each F_i^c is open. By De Morgan, $(\bigcap_i F_i)^c = \bigcup_i F_i^c$, an arbitrary union of open sets, which is open by (a); hence $\bigcap_i F_i$ is closed. Likewise, for finitely many closed sets $F_1, \dots, F_n$, $(\bigcup_{k=1}^n F_k)^c = \bigcap_{k=1}^n F_k^c$ is a finite intersection of open sets, open by (b); hence $\bigcup_{k=1}^n F_k$ is closed.

Question 3

Let $A \subseteq \mathbb{R}$ be non-empty and let $s \in \mathbb{R}$ be an upper bound of A . Prove that $s = \sup A$ if and only if

$$\forall \varepsilon > 0, \exists a \in A \text{ such that } a > s - \varepsilon.$$

SOLUTIONS

Recall that $\sup A$ is the **least** upper bound of A : an upper bound that is $\leq$ every other upper bound. Throughout, only this definition is used — no completeness axiom.

($\Rightarrow$) Suppose $s = \sup A$ and let $\varepsilon > 0$. Since $s - \varepsilon < s$ and s is the *least* upper bound, $s - \varepsilon$ cannot be an upper bound of A . So some $a \in A$ satisfies $a > s - \varepsilon$.

($\Leftarrow$) Suppose s is an upper bound and for every $\varepsilon > 0$ there is $a \in A$ with $a > s - \varepsilon$. Let t be any upper bound of A ; we show $t \geq s$. If instead $t < s$, put $\varepsilon = s - t > 0$. By hypothesis there is $a \in A$ with $a > s - \varepsilon = t$, contradicting that t is an upper bound. Hence every upper bound t satisfies $t \geq s$, so s is the least upper bound: $s = \sup A$.

Question 4

Using the definition **compact** = **closed and bounded** (in $\mathbb{R}^n$), justify each claim.

- $[0, 1]^n$ is compact.
- $(0, 1)$ is not compact.
- $\mathbb{R}$ is not compact.
- The budget set $\{x \in \mathbb{R}^n : x \geq 0, p \cdot x \leq w\}$ is compact, where $p \gg 0$ (all prices strictly positive) and $w > 0$.

SOLUTIONS

- Bounded:** for $x \in [0, 1]^n$ each coordinate lies in $[0, 1]$, so $\|x\| \leq \sqrt{n}$. **Closed:** if $(x_k) \subseteq [0, 1]^n$ and $x_k \rightarrow x$, each coordinate satisfies $0 \leq x_{k,i} \leq 1$, so the order limit theorem gives $0 \leq x_i \leq 1$; thus $x \in [0, 1]^n$. By the sequential characterisation, $[0, 1]^n$ is closed. Closed and bounded, hence compact.
- $(0, 1)$ is bounded but not closed: $x_k = 1/k \in (0, 1)$ and $x_k \rightarrow 0 \notin (0, 1)$, so it does not contain the limit point 0. Not closed $\Rightarrow$ not compact.
- $\mathbb{R}$ is closed but not bounded: for any $M > 0$, the point $M + 1$ satisfies $|M + 1| > M$, so no single M bounds every element. Not bounded $\Rightarrow$ not compact.
- Write $K = \{x : x \geq 0, p \cdot x \leq w\}$. **Bounded:** let $\underline{p} = \min_i p_i > 0$. For $x \in K$ each $x_i \geq 0$, so $\underline{p} \sum_i x_i \leq \sum_i p_i x_i = p \cdot x \leq w$, giving $0 \leq x_i \leq w/\underline{p}$ for every i ; hence $\|x\| \leq \sqrt{n} w/\underline{p}$. **Closed:** each set $\{x : x_i \geq 0\}$ and $\{x : p \cdot x \leq w\}$ is closed (if $x_k \rightarrow x$ with $x_{k,i} \geq 0$ then $x_i \geq 0$, and if $p \cdot x_k \leq w$ then $p \cdot x \leq w$, both by the order limit theorem). K is a finite intersection of these closed sets, hence closed by Question 2(c). Closed and bounded, hence compact.

Question 5

Prove that if a sequence (x_k) in $\mathbb{R}^n$ converges, then its limit is unique.

SOLUTIONS

Suppose $x_k \rightarrow L$ and $x_k \rightarrow L'$; we show $L = L'$. Let $\varepsilon > 0$. By convergence there are K_1, K_2 with $\|x_k - L\| < \varepsilon/2$ for $k \geq K_1$ and $\|x_k - L'\| < \varepsilon/2$ for $k \geq K_2$. For any $k \geq \max\{K_1, K_2\}$, the triangle inequality gives

$$\|L - L'\| \leq \|L - x_k\| + \|x_k - L'\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\|L - L'\| < \varepsilon$ for every $\varepsilon > 0$, which forces $\|L - L'\| = 0$, i.e. $L = L'$.

Question 6

Let $x_k \rightarrow x$ and $y_k \rightarrow y$ in $\mathbb{R}$. Prove:

- (Algebraic Limit Theorem)** $x_k + y_k \rightarrow x + y$; $cx_k \rightarrow cx$; $x_k y_k \rightarrow xy$; and, if $y \neq 0$ and $y_k \neq 0$, $x_k/y_k \rightarrow x/y$.
- (Order Limit Theorem)** if $x_k \leq y_k$ for all k , then $x \leq y$.
- (Squeeze Theorem)** if $x_k \leq z_k \leq y_k$ for all k and $x_k, y_k \rightarrow L$, then $z_k \rightarrow L$.

The quotient limit in (a) is the most delicate part.

SOLUTIONS

- Sum.** Given $\varepsilon > 0$, choose K with $|x_k - x| < \varepsilon/2$ and $|y_k - y| < \varepsilon/2$ for $k \geq K$. Then $|(x_k + y_k) - (x + y)| \leq |x_k - x| + |y_k - y| < \varepsilon$.

Scalar. If $c = 0$ this is trivial. For $c \neq 0$, given $\varepsilon > 0$ choose K with $|x_k - x| < \varepsilon/|c|$; then $|cx_k - cx| = |c||x_k - x| < \varepsilon$.

Product. A convergent sequence is bounded, so there is $M > 0$ with $|x_k| \leq M$ for all k . Then

$$|x_k y_k - xy| = |x_k y_k - x_k y + x_k y - xy| \leq |x_k| |y_k - y| + |y| |x_k - x| \leq M |y_k - y| + |y| |x_k - x|.$$

Given $\varepsilon > 0$, choose K so that $|y_k - y| < \frac{\varepsilon}{2M}$ and $|x_k - x| < \frac{\varepsilon}{2(|y| + 1)}$ for $k \geq K$; then $|x_k y_k - xy| < \varepsilon$.

Quotient. It suffices to show $1/y_k \rightarrow 1/y$ and then apply the product rule. Since $y \neq 0$, taking $\varepsilon_0 = |y|/2$ gives K_0 with $|y_k - y| < |y|/2$ for $k \geq K_0$, so $|y_k| \geq |y| - |y_k - y| > |y|/2$ for such k . Then for $k \geq K_0$,

$$\left| \frac{1}{y_k} - \frac{1}{y} \right| = \frac{|y - y_k|}{|y_k| |y|} \leq \frac{|y - y_k|}{(|y|/2) |y|} = \frac{2}{|y|^2} |y_k - y|.$$

Given $\varepsilon > 0$, also require $|y_k - y| < \frac{|y|^2}{2} \varepsilon$; then $|1/y_k - 1/y| < \varepsilon$. Hence $1/y_k \rightarrow 1/y$, and $x_k/y_k = x_k \cdot (1/y_k) \rightarrow x \cdot (1/y) = x/y$ by the product rule.

- Suppose for contradiction $x > y$, and set $\varepsilon = (x - y)/2 > 0$. For large k , $|x_k - x| < \varepsilon$ and $|y_k - y| < \varepsilon$, so $x_k > x - \varepsilon = \frac{x+y}{2}$ and $y_k < y + \varepsilon = \frac{x+y}{2}$, giving $y_k < x_k$ — contradicting $x_k \leq y_k$. Hence $x \leq y$.
- Given $\varepsilon > 0$, choose K with $|x_k - L| < \varepsilon$ and $|y_k - L| < \varepsilon$ for $k \geq K$, i.e. $L - \varepsilon < x_k$ and $y_k < L + \varepsilon$. Then $L - \varepsilon < x_k \leq z_k \leq y_k < L + \varepsilon$, so $|z_k - L| < \varepsilon$. Hence $z_k \rightarrow L$.

Question 7

- Prove the **Monotone Convergence Theorem**: every bounded, monotone sequence in $\mathbb{R}$ converges. (Prove the non-decreasing case; the non-increasing case is symmetric.)
- Show **directly** from the ε - K definition — without invoking the MCT — that $x_k = 1 - \frac{1}{k}$ converges to 1.

SOLUTIONS

- Let (x_k) be non-decreasing and bounded above. The set $A = \{x_k : k \in \mathbb{N}\}$ is non-empty and bounded above, so by the completeness axiom $s = \sup A$ exists. We claim $x_k \rightarrow s$. Let $\varepsilon > 0$. By the ε -characterisation of the supremum (Question 3), there is an index K with $x_K > s - \varepsilon$. Since (x_k) is non-decreasing, $x_k \geq x_K > s - \varepsilon$ for all $k \geq K$; and $x_k \leq s$ because s is an upper bound. Hence $s - \varepsilon < x_k \leq s < s + \varepsilon$, i.e. $|x_k - s| < \varepsilon$ for $k \geq K$. Therefore $x_k \rightarrow s = \sup_k x_k$. (For a non-increasing sequence bounded below, apply this to $(-x_k)$, giving convergence to $\inf_k x_k$.)
- Let $\varepsilon > 0$. We want $|x_k - 1| = |(1 - \frac{1}{k}) - 1| = \frac{1}{k} < \varepsilon$. By the Archimedean property, choose $K \in \mathbb{N}$ with $K > 1/\varepsilon$. Then for all $k \geq K$, $\frac{1}{k} \leq \frac{1}{K} < \varepsilon$, so $|x_k - 1| < \varepsilon$. Hence $x_k \rightarrow 1$.

Question 8

- Prove the **sequential characterisation of closed sets**: $F \subseteq \mathbb{R}^n$ is closed if and only if every convergent sequence $(x_k) \subseteq F$ has its limit in F .
- Use it to show $[0, 1]$ is closed and $(0, 1)$ is not.

SOLUTIONS

- Recall F is closed iff its complement F^c is open.
 $(\Rightarrow)$ Suppose F is closed. Let $(x_k) \subseteq F$ with $x_k \rightarrow x$; we show $x \in F$. If instead $x \in F^c$, then since F^c is open there is $\varepsilon > 0$ with $B(x, \varepsilon) \subseteq F^c$. But $x_k \rightarrow x$ yields some k with $\|x_k - x\| < \varepsilon$, i.e. $x_k \in B(x, \varepsilon) \subseteq F^c$, contradicting $x_k \in F$. Hence $x \in F$.
 $(\Leftarrow)$ Suppose every convergent sequence in F has its limit in F . We show F^c is open. If not, some $x \in F^c$ has no ball contained in F^c : for every $\varepsilon > 0$, $B(x, \varepsilon)$ meets F . Taking $\varepsilon = 1/k$, pick $x_k \in F \cap B(x, 1/k)$. Then $\|x_k - x\| < 1/k \rightarrow 0$, so $x_k \rightarrow x$ with $(x_k) \subseteq F$; by hypothesis $x \in F$, contradicting $x \in F^c$. Hence F^c is open and F is closed.
- $[0, 1]$ closed:** if $(x_k) \subseteq [0, 1]$ and $x_k \rightarrow x$, then $0 \leq x_k \leq 1$ gives $0 \leq x \leq 1$ by the order limit theorem, so $x \in [0, 1]$; by (a), $[0, 1]$ is closed. **$(0, 1)$ not closed:**

$x_k = 1/k \in (0, 1)$ and $x_k \rightarrow 0 \notin (0, 1)$, so (a) fails and $(0, 1)$ is not closed.

Question 9

- Prove that if $x_k \rightarrow L$, then every subsequence (x_{k_j}) converges to L .
- (Challenging.) Prove the **Bolzano–Weierstrass theorem**: every bounded sequence in $\mathbb{R}$ has a convergent subsequence. *Hint: first show every real sequence has a monotone subsequence, then apply the Monotone Convergence Theorem.*

SOLUTIONS

- Let $\varepsilon > 0$. Since $x_k \rightarrow L$, there is K with $|x_k - L| < \varepsilon$ for all $k \geq K$. The subsequence indices satisfy $k_1 < k_2 < \dots$, so $k_j \geq j$; thus $j \geq K \Rightarrow k_j \geq K \Rightarrow |x_{k_j} - L| < \varepsilon$. Hence $x_{k_j} \rightarrow L$.
- Step 1 — every sequence has a monotone subsequence.** Call an index m a *peak* if $x_m \geq x_k$ for all $k > m$.
 - If there are infinitely many peaks $m_1 < m_2 < \dots$, then by definition of peak $x_{m_1} \geq x_{m_2} \geq \dots$, a non-increasing subsequence.
 - If there are only finitely many peaks, choose k_1 larger than every peak. Since k_1 is not a peak, there is $k_2 > k_1$ with $x_{k_2} > x_{k_1}$. Since k_2 is also past every peak, it is not a peak, so there is $k_3 > k_2$ with $x_{k_3} > x_{k_2}$. Continuing, we obtain a strictly increasing subsequence $x_{k_1} < x_{k_2} < \dots$.

Either way we get a monotone subsequence.

Step 2. Let (x_k) be bounded. By Step 1 it has a monotone subsequence (x_{k_j}) , which is bounded (being a subsequence of a bounded sequence). By the Monotone Convergence Theorem, (x_{k_j}) converges. Hence (x_k) has a convergent subsequence.

(In $\mathbb{R}^n$: extract a subsequence convergent in the first coordinate, then a further subsequence convergent in the second, and so on; after n steps the final subsequence converges in every coordinate, hence in $\mathbb{R}^n$.)

Question 10

Let $f : X \rightarrow Y$ and $A, B \subseteq Y$. Prove the preimage identities:

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), \quad f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B), \quad f^{-1}(A^c) = (f^{-1}(A))^c.$$

SOLUTIONS

All three follow by unwinding the definition $f^{-1}(S) = \{x : f(x) \in S\}$ and the matching logical connective.

Union. $x \in f^{-1}(A \cup B) \iff f(x) \in A \cup B \iff (f(x) \in A \text{ or } f(x) \in B) \iff x \in f^{-1}(A) \cup f^{-1}(B)$.

Intersection. The same with “or” replaced by “and”: $x \in f^{-1}(A \cap B) \iff (f(x) \in A \text{ and } f(x) \in B) \iff x \in f^{-1}(A) \cap f^{-1}(B)$.

Complement. $x \in f^{-1}(A^c) \iff f(x) \in A^c \iff f(x) \notin A \iff x \notin f^{-1}(A) \iff x \in (f^{-1}(A))^c$. (Here A^c is taken in Y and $(f^{-1}(A))^c$ in X .)

Question 11

Prove that the ε - δ and sequential definitions of continuity at a point are equivalent: for $f : X \rightarrow \mathbb{R}^m$ and $x_0 \in X$,

$$(\forall \varepsilon > 0 \exists \delta > 0 : \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon) \iff (x_k \rightarrow x_0 \Rightarrow f(x_k) \rightarrow f(x_0)).$$

SOLUTIONS

($\Rightarrow$) Assume the ε - δ condition and let $x_k \rightarrow x_0$. Fix $\varepsilon > 0$ and take the corresponding $\delta > 0$. Since $x_k \rightarrow x_0$, there is K with $\|x_k - x_0\| < \delta$ for $k \geq K$; then $\|f(x_k) - f(x_0)\| < \varepsilon$ for $k \geq K$. Hence $f(x_k) \rightarrow f(x_0)$.

($\Leftarrow$) We prove the contrapositive: if ε - δ fails, so does the sequential condition. If ε - δ fails, there is some $\varepsilon_0 > 0$ such that for every $\delta > 0$ some point within δ of x_0 has image at distance $\geq \varepsilon_0$ from $f(x_0)$. Taking $\delta = 1/k$, choose $x_k \in X$ with $\|x_k - x_0\| < 1/k$ but $\|f(x_k) - f(x_0)\| \geq \varepsilon_0$. Then $x_k \rightarrow x_0$ while $f(x_k) \not\rightarrow f(x_0)$, so the sequential condition fails. By contraposition, the sequential condition implies ε - δ .

Question 12

Let $f, g : X \rightarrow \mathbb{R}$ be continuous at x_0 , and let $c \in \mathbb{R}$.

- a. Using the sequential characterisation of continuity, prove that $f + g$, cf , and fg are continuous at x_0 , and that f/g is continuous at x_0 provided $g(x_0) \neq 0$.
- b. Prove that a composition of continuous functions is continuous: if $f : X \rightarrow Y$ is continuous at x_0 and $g : Y \rightarrow Z$ is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0 .

SOLUTIONS

- a. Let $x_k \rightarrow x_0$ in X . By continuity (sequential form), $f(x_k) \rightarrow f(x_0)$ and $g(x_k) \rightarrow g(x_0)$. By the Algebraic Limit Theorem (Question 6),

$$f(x_k) + g(x_k) \rightarrow f(x_0) + g(x_0), \quad c f(x_k) \rightarrow c f(x_0), \quad f(x_k)g(x_k) \rightarrow f(x_0)g(x_0),$$

and, when $g(x_0) \neq 0$, $f(x_k)/g(x_k) \rightarrow f(x_0)/g(x_0)$ (continuity gives $g(x_k) \rightarrow g(x_0) \neq 0$, so $g(x_k) \neq 0$ for large k and the quotient rule applies). Since $x_k \rightarrow x_0$ was arbitrary, each of $f + g$, cf , fg , f/g satisfies the sequential criterion, hence is continuous at x_0 .

- b. Let $x_k \rightarrow x_0$. Since f is continuous at x_0 , $f(x_k) \rightarrow f(x_0)$. Applying the sequential criterion for g at $f(x_0)$ to the sequence $y_k := f(x_k) \rightarrow f(x_0)$ gives $g(y_k) \rightarrow g(f(x_0))$, i.e. $(g \circ f)(x_k) \rightarrow (g \circ f)(x_0)$. Hence $g \circ f$ is continuous at x_0 .

Question 13

(Challenging.)

- a. Prove the **Extreme Value Theorem**: if $K \subseteq \mathbb{R}^n$ is compact and $f : K \rightarrow \mathbb{R}$ is continuous, then f attains a maximum and a minimum on K .
- b. Prove the **Intermediate Value Theorem**: if $f : [a, b] \rightarrow \mathbb{R}$ is continuous and y lies between $f(a)$ and $f(b)$, then $f(c) = y$ for some $c \in [a, b]$.

SOLUTIONS

- a. We show f attains a maximum (for the minimum, apply the result to $-f$).

Bounded above. Suppose not: then for each k there is $x_k \in K$ with $f(x_k) > k$. Since K is bounded, (x_k) is bounded, so by Bolzano–Weierstrass a subsequence $x_{k_j} \rightarrow x^*$; and $x^* \in K$ because K is closed. By continuity $f(x_{k_j}) \rightarrow f(x^*)$, a finite number — but $f(x_{k_j}) > k_j \rightarrow \infty$, a contradiction. So f is bounded above, and $M := \sup_{x \in K} f(x)$ exists by completeness.

Attainment. By the ε -characterisation of the supremum (Question 3), for each k choose $z_k \in K$ with $f(z_k) > M - \frac{1}{k}$; with $f(z_k) \leq M$ this gives $f(z_k) \rightarrow M$. By Bolzano–Weierstrass and closedness, some subsequence $z_{k_j} \rightarrow x^* \in K$, and continuity gives $f(z_{k_j}) \rightarrow f(x^*)$. Since also $f(z_{k_j}) \rightarrow M$, uniqueness of limits gives $f(x^*) = M$. Thus f attains its maximum at x^* .

- b. Assume without loss of generality $f(a) < y < f(b)$ (if $f(a) > f(b)$, apply the argument to $-f$ and $-y$; if y equals an endpoint value there is nothing to prove).

Let

$$S = \{x \in [a, b] : f(x) < y\}.$$

S is non-empty ($a \in S$) and bounded above by b , so $c = \sup S$ exists by completeness, with $c \in [a, b]$. We rule out both strict inequalities using continuity at c .

- If $f(c) < y$: by continuity there is $\delta > 0$ with $f(x) < y$ for all $x \in [a, b]$ within δ of c . Since $f(b) > y$ we have $c < b$, so points just to the right of c belong to S , contradicting $c = \sup S$.
- If $f(c) > y$: then $c \neq a$ (as $f(a) < y$), so $c > a$; by continuity there is $\delta > 0$ with $f(x) > y$ for x within δ of c . No point of $(c - \delta, c]$ lies in S , so $c - \delta$ is an upper bound of S smaller than c , contradicting $c = \sup S$.

Both cases are impossible, so $f(c) = y$ (and $c \in (a, b)$ since $f(c) = y$ differs from $f(a)$ and $f(b)$).