

More on Optimization

UBC Economics Math Camps

Vitor Farinha Luz

Viktoriya Hnatkovska

Ying Gao



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Concavity: second-order conditions



Derivative characterization — multiple variables

- A function of n variables has n^2 possible second derivatives: $\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial \left(\frac{\partial f(x)}{\partial x_i} \right)}{\partial x_j}$
- Hessian matrix collects them: $H(x) = \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right]_{i,j}$
- A function f is concave if, and only if, $H(x)$ is negative semi-definite for all values x
- If $H(\cdot)$ is negative definite then f is strictly concave.
 - We will cover negative semi-definiteness in ECON526



Optimization with equality constraints



Lagrangian as a Bordered Hessian

The first-order conditions are:

$$\Lambda_x = f_x(x^*, y^*) - \lambda^* g_x(x^*, y^*) = 0.$$

$$\Lambda_y = f_y(x^*, y^*) - \lambda^* g_y(x^*, y^*) = 0.$$

$$\Lambda_\lambda = c - g(x^*, y^*) = 0.$$

These are the necessary conditions for the optimum. To derive sufficient conditions, we define a bordered Hessian:

$$H = \begin{bmatrix} 0 & g_x & g_y \\ g_x & f_{xx} - \lambda g_{xx} & f_{xy} - \lambda g_{xy} \\ g_y & f_{xy} - \lambda g_{xy} & f_{yy} - \lambda g_{yy} \end{bmatrix}$$



Bordered Hessian augments the regular Hessian matrix of the objective function with the gradients of the constraint functions.

- When the first-order conditions above hold, then
 - if bordered Hessian is negative semi-definite, (x^*, y^*) is a constrained local maximum
 - if bordered Hessian is positive semi-definite, (x^*, y^*) is a constrained local minimum
- Combining the first-order conditions above and assuming that the derivatives are not zero, we get

$$\frac{f_x(x^*, y^*)}{f_y(x^*, y^*)} = \frac{g_x(x^*, y^*)}{g_y(x^*, y^*)}$$



Relationship with the unconstrained optimization problem

The first-order conditions and second-order conditions in this constrained problem are closely related to the corresponding conditions in the unconstrained problem studied earlier.

To show this, notice that the constraint $g(\mathbf{x}, \mathbf{y}) = c$ can be re-arranged to write $\mathbf{y}$ as a function of $\mathbf{x}$, say

$$\mathbf{y} = h(\mathbf{x})$$

Then, by substituting it into the objective function, the objective function becomes

$$f(\mathbf{x}, h(\mathbf{x}))$$



- The original problem with two choice variables and a constraint has been reduced to a one-variable unconstrained problem
- Maximizing $f(x, h(x))$ with respect to x , the first-order condition is

$$0 = f_x + h' f_y$$

where $h' = \frac{dy}{dx} = -\frac{g_x}{g_y}$

- Substitute this into the first-order condition to get

$$0 = f_x - \frac{g_x}{g_y} f_y$$

- Collecting the terms gives:



$$\frac{f_x}{f_y} = \frac{g_x}{g_y},$$

which is the same condition obtained under the Lagrangian. Together with the constraint $g(x, y) = c$ it can be solved for (x^*, y^*) .

- In the same spirit, one can also show the equivalence of the second-order conditions between this unconstrained formulation and the constrained problem, but we will omit it here.



Interpretation of the Lagrange multiplier

The Lagrange multiplier measures the marginal impact on the value of the objective due to changing the constraint “a little” (marginally).

To see this, consider the change in the objective function f when the constraint c changes: df/dc .

$$\frac{df}{dc} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial c} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial c}$$

- From the first-order conditions we have:



$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} = \lambda g_x$$
$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} = \lambda g_y$$

- If we totally differentiate the constraint with respect to c , we get

$$\frac{dg}{dc} = g_x \frac{\partial x}{\partial c} + g_y \frac{\partial y}{\partial c} = 1$$

- Substituting this into df/dc gives

$$\frac{df}{dc} = \lambda$$



Useful references

- Bergin (2015)
- Simon and Blume (1994)



References

- Bergin, James. 2015. *Mathematics for Economists with Applications*. Routledge. <https://doi.org/10.4324/9781315819532>.
- Simon, Carl P., and Lawrence Blume. 1994. *Mathematics for Economists*. New York: W. W. Norton & Company.

