

Proof and Real Analysis

UBC Economics Math Camps

William Duan



Table of contents

- Logic and Proof Techniques
- Set Theory
- Sequences and Limits
- Functions and Continuity
- Summary



Logic and Proof Techniques



Mathematical Statements

Definition — Statement (Proposition)

A **statement** is a declarative sentence that is either true or false, but not both.

Statements:

- “It is raining.” (*true or false*)
- $2 + 2 = 4$ (*true*)
- $\sqrt{2} \in \mathbb{Q}$ (*false*)

Not statements:

- $x > 0$ (*depends on x*)
- “Is this true?” (*a question*)

Quantifiers

Much of mathematics involves “For all...” or “There exists...” — written $\forall$ and $\exists$.



Logical Connectives and Truth Tables

Let P = "It is raining" and Q = "It is cold."

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$
T	T	F	T	T	T	$\top^+$
T	F	F	F	T	F	F
F	T	T	F	T	$\top^+$	$\top^+$
F	F	T	F	F	$\top^+$	T

- $\neg P$: **negation**; $P \wedge Q$: **conjunction**; $P \vee Q$: **disjunction**
- Entries marked $\dagger$ are *vacuously true*: $P \Rightarrow Q$ is vacuous when P is false (rows 3–4); $\neg Q \Rightarrow \neg P$ is vacuous when Q is true (rows 1, 3)
- The two implication columns are identical in every row – $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$



Implication: $P \Rightarrow Q$

Vocabulary

In $P \Rightarrow Q$, P is the **hypothesis** and Q is the **conclusion**.

With P = "It is raining," Q = "It is cold":

- **Original** $P \Rightarrow Q$: "If it is raining, then it is cold."
- **Contrapositive** $\neg Q \Rightarrow \neg P$: "If it is *not* cold, then it is *not* raining." (*equivalent*)
- **Converse** $Q \Rightarrow P$: "If it is cold, then it is raining." (*not equivalent*)

A common mistake: $P \nRightarrow Q$ is not the same as $P \Rightarrow \neg Q$

- $P \nRightarrow Q$: "It is *not* the case that raining implies cold" — the link fails somewhere
- $P \Rightarrow \neg Q$: "If it is raining, then it is *not* cold" — a positive claim in the other direction



What Does It Mean to Prove $P \Rightarrow Q$?

Proving $P \Rightarrow Q$ means: **assuming P is true, we can never be in row 2.**

P	Q	$P \Rightarrow Q$	
T	T	T	← Direct proof : show we always land here
T	F	F	← must rule this out
F	T	T	(vacuously true)
F	F	T	(vacuously true)

Three logically distinct ways to rule out row 2:

1. Show P true always forces Q true — *direct*
2. Prove the equivalent $\neg Q \Rightarrow \neg P$ — *contrapositive*
3. Assume P true and Q false, derive a contradiction — *contradiction*



Three Proof Strategies for $P \Rightarrow Q$

Direct Proof — stay in row 1

Assume P is true. Derive Q by a chain of valid steps.

Example: "It is raining $\Rightarrow$ the ground is wet $\Rightarrow$ it is slippery $\Rightarrow$..."

Proof by Contrapositive — prove $\neg Q \Rightarrow \neg P$

Assume Q is false. Derive P is false.

Example: "It is *not* cold $\Rightarrow$ it is *not* raining."

Proof by Contradiction — row 2 leads to a falsehood

Assume P true *and* Q false. Derive a logical impossibility.

Example: "Suppose it is raining but not cold. Then [contradiction]."



Quantified Statements and Implications

Key observation

Universal statements and implications are two sides of the same coin.

$\forall$ Statement

"Any day that is raining is cold."

Being cold is a property shared by all rainy days.

Implication Statement

"If it is raining, then it is cold."

Being cold is true whenever it is raining is true.

Proving $P \Rightarrow Q$ is always equivalent to showing that Q holds for **every** instance where P holds.



Biconditionals

Definition – Biconditional

$P \Leftrightarrow Q$ (“ P if and only if Q ”, abbreviated **iff**) means:

$$P \Rightarrow Q \quad \text{and} \quad Q \Rightarrow P.$$

Proof strategy

Prove both directions separately. Label them ($\Rightarrow$) and ($\Leftarrow$) clearly, and apply whichever technique suits each direction.

Notation reminder

- “ P only if Q ” means $P \Rightarrow Q$
- “ P if Q ” means $Q \Rightarrow P$
- “ P if and only if Q ” combines both



Set Theory



Sets and Basic Operations

Definition – Set

A **set** is a collection of distinct objects, called **elements**. We write $x \in A$ if x belongs to A , and $A \subseteq B$ if every element of A is in B .

Operations:

- $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- $A \setminus B = \{x : x \in A, x \notin B\}$
- $A^c = \{x \in X : x \notin A\}$

De Morgan's Laws:

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

Definition – Open Ball

The **open ball** of radius ε centred at $x \in \mathbb{R}^n$:

$$B(x, \varepsilon) = \{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\}$$

► Worked solution



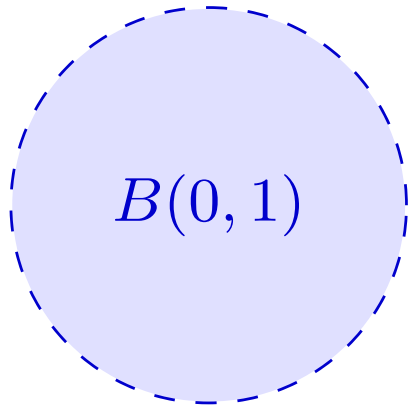
Open and Closed Sets

Definition — Open Set

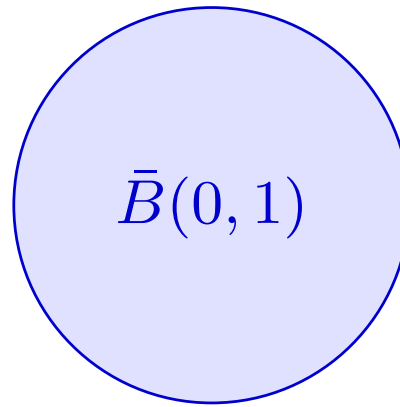
$U \subseteq \mathbb{R}^n$ is **open** if $\forall x \in U, \exists \varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$.

Definition — Closed Set

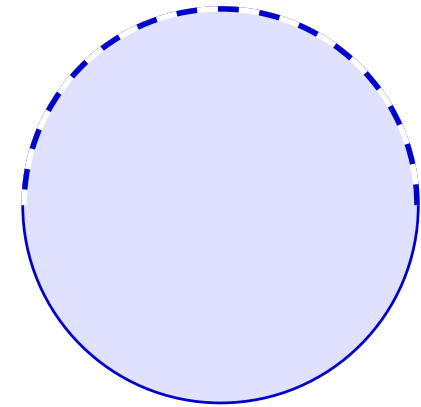
$F \subseteq \mathbb{R}^n$ is **closed** if its complement F^c is open.



Open



Closed



Neither

“Not open” $\neq$ “closed.” $\mathbb{R}^n$ and $\emptyset$ are *both* open and closed.



Properties of Open and Closed Sets

Open sets are preserved under...

- **Arbitrary** unions
- **Finite** intersections

Warning: infinite intersections can fail.

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

is closed.

Closed sets are preserved under...

- **Arbitrary** intersections
- **Finite** unions

Warning: infinite unions can fail.

$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = (0, 1]$$

is not closed.

► Worked solution



The Real Line: Intervals and Completeness

Intervals in $\mathbb{R}$:

- (a, b) – open
- $[a, b]$ – closed
- $[a, b), (a, b]$ – neither

Balls in $\mathbb{R}^n$:

- $B(x, r) = \{y : \|y - x\| < r\}$ – open
- $\bar{B}(x, r) = \{y : \|y - x\| \leq r\}$ – closed

Definition – Supremum / Infimum

Let $A \subseteq \mathbb{R}$ be non-empty.

$\sup A$ = **least upper bound** of A .

$\inf A$ = **greatest lower bound** of A .

Completeness Axiom

Every non-empty $A \subseteq \mathbb{R}$ bounded above has a supremum in $\mathbb{R}$.

► Worked solution



Bounded and Compact Sets

Definition – Bounded Set

$A \subseteq \mathbb{R}^n$ is **bounded** if $\exists M > 0$ such that $\|x\| \leq M$ for all $x \in A$.

Definition – Compact Set (in $\mathbb{R}^n$)

$K \subseteq \mathbb{R}^n$ is **compact** if it is both **closed** and **bounded**.

Example

- $[0,1]^n$ is compact.
- $(0,1)$ is not compact – it is not closed.
- $\mathbb{R}$ is not compact – it is not bounded.
- The budget set $\{x \geq 0 : p \cdot x \leq w\}$ is compact.

► Worked solution



Sequences and Limits



Sequences

Definition – Sequence

A **sequence** in $\mathbb{R}^n$ is a function $\mathbb{N} \rightarrow \mathbb{R}^n$. We write $(x_k)_{k=1}^{\infty}$ or (x_k) .

Definition – Convergence

(x_k) **converges** to $L \in \mathbb{R}^n$, written $x_k \rightarrow L$, if:

$$\forall \varepsilon > 0, \quad \exists K \in \mathbb{N} \text{ such that } k \geq K \Rightarrow \|x_k - L\| < \varepsilon.$$

Proposition – Uniqueness of Limits

If (x_k) converges, its limit is unique.

▶ Worked solution



Properties of Convergent Sequences

Let $(x_k) \rightarrow x$ and $(y_k) \rightarrow y$ in $\mathbb{R}$.

Algebraic Limit Theorem

1. $x_k + y_k \rightarrow x + y$
2. $c x_k \rightarrow c x$
3. $x_k y_k \rightarrow x y$
4. $x_k / y_k \rightarrow x / y$ (provided $y \neq 0, y_k \neq 0$)

Order Limit Theorem

If $x_k \leq y_k$ for all k , then $x \leq y$.

Squeeze Theorem

If $x_k \leq z_k \leq y_k$ for all k and $x_k, y_k \rightarrow L$, then $z_k \rightarrow L$.

► Worked solution



Monotone Sequences

Definition — Monotone Sequence

(x_k) is **non-decreasing** if $x_k \leq x_{k+1}$ for all k , and **non-increasing** if $x_k \geq x_{k+1}$ for all k .

Theorem — Monotone Convergence Theorem

Every bounded monotone sequence in $\mathbb{R}$ converges.

- If (x_k) is non-decreasing and bounded above, then $x_k \rightarrow \sup_k x_k$.
- If (x_k) is non-increasing and bounded below, then $x_k \rightarrow \inf_k x_k$.

Example

$x_k = 1 - \frac{1}{k}$ is non-decreasing and bounded above by **1**, so it converges. Indeed $x_k \rightarrow 1 = \sup_k x_k$.

► Worked solution



Limit Points and Closed Sets

Definition – Limit Point

$x \in \mathbb{R}^n$ is a **limit point** of $A \subseteq \mathbb{R}^n$ if there exists a sequence (x_k) in A with $x_k \neq x$ for all k and $x_k \rightarrow x$.

Theorem – Sequential characterisation of closed sets

$F \subseteq \mathbb{R}^n$ is closed $\iff F$ contains all its limit points:

$(x_k) \subseteq F$ and $x_k \rightarrow x \implies x \in F$.

Example

- $[0,1]$: any convergent sequence in $[0,1]$ has its limit in $[0,1]$. *Closed*.
- $(0,1)$: $x_k = \frac{1}{k} \in (0,1)$ but $x_k \rightarrow 0 \notin (0,1)$. *Not closed*.

► Worked solution



Subsequences and Accumulation Points

Definition – Subsequence

A **subsequence** of (x_k) is (x_{k_j}) where $k_1 < k_2 < k_3 < \dots$.

Proposition

If $x_k \rightarrow L$, then every subsequence converges to L .

Definition – Accumulation Point

L is an **accumulation point** of (x_k) if some subsequence converges to L .

Theorem – Bolzano–Weierstrass

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

► Worked solution



Functions and Continuity



Functions

Definition – Function

A **function** $f : X \rightarrow Y$ assigns to each $x \in X$ exactly one $f(x) \in Y$.

- X : **domain**; Y : **codomain**
- **Range**: $f(X) = \{f(x) : x \in X\} \subseteq Y$

Preimage

For $B \subseteq Y$:

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

Preimage identities

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

$$f^{-1}(A^c) = (f^{-1}(A))^c$$

► Worked solution



Continuity

Definition – ε - δ (pointwise)

$f : X \rightarrow \mathbb{R}^m$ is **continuous at** x_0 if:

$$\forall \varepsilon > 0, \quad \exists \delta > 0 : \quad \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon.$$

Sequential characterisation (equivalent)

f is continuous at x_0 if and only if:

$$x_k \rightarrow x_0 \Rightarrow f(x_k) \rightarrow f(x_0).$$

Topological characterisation (equivalent)

f is continuous on X if and only if $f^{-1}(V)$ is open for every open $V \subseteq \mathbb{R}^m$.

► Worked solution



Algebra of Continuous Functions

Proposition

If $f, g : X \rightarrow \mathbb{R}$ are continuous at x_0 , then so are:

- $f + g$, cf (any constant c), fg
- f/g provided $g(x_0) \neq 0$

Proposition – Composition

If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then $g \circ f$ is continuous.

Example

Polynomials are continuous everywhere. Rational functions are continuous wherever the denominator is non-zero.

► Worked solution



Key Theorems

Theorem — Extreme Value Theorem

Let $f : K \rightarrow \mathbb{R}$ be continuous with $K \subseteq \mathbb{R}^n$ **compact**. Then f attains its maximum and minimum on K :

$$\exists x^*, x_* \in K : \quad f(x_*) \leq f(x) \leq f(x^*) \quad \forall x \in K.$$

Theorem — Intermediate Value Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. If y lies between $f(a)$ and $f(b)$, then $\exists c \in (a, b)$ with $f(c) = y$.

EVT in economics: the budget set $B(p, w)$ is compact and utility is continuous, so a utility maximiser always exists.

► Worked solution



Summary



Summary

Logic & Proofs

- Statements, quantifiers, connectives
- $\forall x, P(x)$ as an implication
- Contrapositive $\equiv$ original; converse $\not\equiv$
- Direct / contrapositive / contradiction
- Biconditionals: prove both directions

Set Theory

- Operations, De Morgan's laws
- Open / closed sets and properties
- Compact = closed + bounded

Sequences

- ε - K definition of convergence
- Algebraic, order, squeeze theorems
- Bounded monotone $\Rightarrow$ converges
- Closed sets contain all limit points
- Bolzano–Weierstrass

Functions & Continuity

- Three equivalent characterisations
- Algebra of continuous functions
- Extreme Value Theorem
- Intermediate Value Theorem

